

# HYPERSURFACE-INVARIANT APPROACH TO COSMOLOGICAL PERTURBATIONS

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## Abstract

Using Hamilton-Jacobi theory, we develop a formalism for solving semi-classical cosmological perturbations which does not require an explicit choice of time-hypersurface. The Hamilton-Jacobi equation for gravity interacting with matter (either a scalar or dust field) is solved by making an Ansatz which includes all terms quadratic in the spatial curvature. Gravitational radiation and scalar perturbations are treated on an equal footing. Our technique encompasses linear perturbation theory and it also describes some mild nonlinear effects. As a concrete example of the method, we compute the galaxy-galaxy correlation function as well as large-angle microwave background fluctuations for power-law inflation, and we compare with recent observations.

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## I. INTRODUCTION

The freedom in choosing a time-hypersurface in general relativity is sometimes viewed as a curse because it leads to numerical problems, difficulties in quantization, etc.. However, for semi-classical analyses based on the Hamilton-Jacobi (HJ) equation, it is actually a blessing.

Even in classical general relativity, selection of a useful time foliation is often a difficult task. In solving Einstein's equations using the ADM form (see, e.g., ref. [1]), arbitrary choices must be made for the lapse  $N$  and the shift  $N^i$  functions; these fields reflect our liberty in choosing the time hypersurface as well as the spatial coordinates. Hamilton-Jacobi theory provides an elegant method of bypassing these very difficult decisions.

It is remarkable that the Hamilton-Jacobi equation for general relativity refers neither to the lapse nor to the shift functions. As a result, a solution for the generating functional  $\mathcal{S}$  is valid for *all* choices of the temporal and spatial coordinates. The HJ equation is the natural starting point for a hypersurface and gauge-invariant analysis — it yields a covariant formulation. It is analogous to the Tomonaga-Schwinger equation that was applied successfully to quantum electrodynamics [2]. However, solutions to the HJ equation are difficult to obtain because one must solve for the entire ensemble of evolving universes that is described by *superspace*.

In a series of papers [3] – [5], we have developed a systematic method of solving the HJ equation by using an expansion in spatial gradients. Using the first few terms, one can derive the nonlinear Zel'dovich approximation [6] and its higher order corrections which describe the formation of pancake structures in a dust-dominated Universe [7] – [9]. Causality is maintained in the relativistic theory, and the final expressions are actually simpler than those obtained from the Newtonian theory (see, e.g., Moutarde et al [10] and Buchert and Ehlers [11]).

Moreover, Parry, Salopek and Stewart [5] derived a recursion relation which enables one to compute the higher order terms for the generating functional from the previous orders. Two useful techniques were employed: a conformal transformation of the 3-metric as well as

a line-integral in superspace. Deforming the contour of integration corresponds to choosing an alternative time-integration parameter. If the endpoints of integration were fixed, all such contours gave identical results provided the theory was invariant under reparametrizations of the spatial coordinates. Time-reparametrization invariance was closely related to spatial gauge-invariance.

However, in the semi-classical problems of interest to inflationary cosmology, a finite number of terms is insufficient. In this paper, we will consider all terms which are quadratic in the curvature. We will effectively sum an *infinite* number of terms in the spatial gradient expansion. Our procedure is analogous to that developed by Barvinsky and Vilkovisky [12] in a very different context: the one-loop effective action for gravity interacting with matter. In this way, we recover the results of linear perturbation theory considered during the 1982 Nuffield workshop [13] – [17], as well as those of Mukhanov, Feldman and Brandenberger [18]. Our formalism also describes some mildly nonlinear effects which we hope to apply to stochastic inflation [19] – [24]. Although there have been some interesting proposals, the choice of time-hypersurface in stochastic inflation still requires further clarification [22]. For example, Linde *et al* [24] have pointed out that eternal inflation may appear differently on various time-hypersurface slices. A covariant formulation would be advantageous.

In Sec. II, we set forth the HJ equation and the momentum constraint equation. We consider two case of physical interest: (1) a scalar field in an inflationary universe and (2) dust in a matter-dominated epoch, with or without a cosmological constant. One can factor out the effects of the long-wavelength background by using a conformal transformation of the 3-metric. We review the long-wavelength formalism using the concept of *field-space diagrams* which provide a simple illustration of the concept of hypersurface transformation. These diagrams are similar in spirit to Minkowski diagrams that proved useful in understanding special relativity.

In Sec. III, we suggest an Ansatz which is second order in the spatial curvature. Arbitrary coefficients appear that are functions of the matter field and the spatial Laplacian operator. Substitution into the HJ equation lead to two linear differential equations, which describe the

evolution of the scalar modes as well as the tensor modes of the 3-metric. For an inflationary cosmology, we choose initial conditions that are consistent with the Bunch-Davies vacuum [25].

There is growing interest in the gravitational waves produced during inflation because it was pointed out [26] that they could provide a large part of the signal detected by the DMR (Differential Microwave Radiometer) experiment on the Cosmic Background Explorer (COBE) satellite [27], and yet be consistent with structure formation. In our approach, density perturbations and gravitational waves are treated on an equal footing. In Sec. V, we compute large angle microwave background fluctuations for various inflationary models and we compare with COBE's two-year data set [28], [29].

After the ground work had been laid by Dirac (see, e.g., ref. [30]), the Hamilton-Jacobi equation for general relativity was first written down by Peres [31] in 1962. For flat space-time, Kuchař [32] solved for the semi-classical wave-functional describing the ground state. By making explicit gauge choices, Halliwell and Hawking [33] gave approximate results for the wavefunctional during the inflationary epoch. Salopek, Bond and Bardeen [34] quantized the system using a Heisenberg formulation, and they computed the power spectra numerically for numerous models utilizing either one or two scalar fields. By expanding the action to second order in perturbations, Mukhanov *et al* gave an alternative prescription for quantizing this system in arbitrary gauges. Their final results were elegant, but because they perturbed the lapse and shift functions, their method was somewhat tedious. Our method removes this unattractive feature and it generalizes the method of Kuchař and Halliwell & Hawking.

(Units are chosen so that  $c = 8\pi G = 8\pi/m_P^2 = \hbar = 1$ . The sign conventions of Misner, Thorne and Wheeler [1] will be adopted throughout.)

## II. THE HAMILTON-JACOBI EQUATION FOR GENERAL RELATIVITY

For a single scalar field  $\phi$  interacting with gravity, the HJ equation and the momentum constraint equation are:

$$0 = \mathcal{H}(x) = \gamma^{-1/2} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}(x)} [2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)] + \frac{1}{2}\gamma^{-1/2} \left( \frac{\delta \mathcal{S}}{\delta \phi(x)} \right)^2 + \gamma^{1/2} V(\phi(x)) + \left[ -\frac{1}{2}\gamma^{1/2} R + \frac{1}{2}\gamma^{1/2} \gamma^{ij} \phi_{,i} \phi_{,j} \right], \quad (2.1a)$$

$$0 = \mathcal{H}_i(x) = -2 \left( \gamma_{ik} \frac{\delta \mathcal{S}}{\delta \gamma_{kj}(x)} \right)_{,j} + \frac{\delta \mathcal{S}}{\delta \gamma_{kl}(x)} \gamma_{kl,i} + \frac{\delta \mathcal{S}}{\delta \phi(x)} \phi_{,i}. \quad (2.1b)$$

In the ADM formalism, the line element is written as

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = \left( -N^2 + \gamma^{ij} N_i N_j \right) dt^2 + 2N_i dt dx^i + \gamma_{ij} dx^i dx^j, \quad (2.2)$$

where  $N$  and  $N_i$  are the lapse and shift functions respectively, and  $\gamma_{ij}$  is the 3-metric. In eq.(2.1a),  $R$  denotes the Ricci scalar of the 3-metric. The object of chief importance is the generating functional  $\mathcal{S} \equiv \mathcal{S}[\gamma_{ij}(x), \phi(x)]$ . For each universe with field configuration  $[\gamma_{ij}(x), \phi(x)]$ , it assigns a number which can be complex. The generating functional is the ‘phase’ of the wavefunctional in the semi-classical approximation:

$$\Psi \sim e^{i\mathcal{S}}. \quad (2.3)$$

For the applications that we are considering, the prefactor before the exponential is not very important, although it has interesting consequences for quantum cosmology [35]. The probability functional,

$$\mathcal{P} \equiv |\Psi|^2, \quad (2.4)$$

is just the square of the wavefunctional (see, e.g., ref. [36]). It is the focus of attention in cosmology during inflation, and even during the matter-dominated era. The Hamilton-Jacobi equation (2.1a) and the momentum constraint (2.1b) follow, respectively, from the  $G_0^0$  and  $G_i^0$  Einstein equations with the canonical momenta replaced by functional derivatives of  $\mathcal{S}$ :

$$\pi^{ij}(x) = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)}, \quad \text{and} \quad \pi^\phi(x) = \frac{\delta \mathcal{S}}{\delta \phi(x)}. \quad (2.5)$$

Eq.(2.1a) is the relativistic generalization of the Newton-Poisson relation, whereas eq.(2.1b) demands that the generating functional be invariant under an arbitrary change of spatial coordinates (see, e.g., Misner *et al* [1], p.1185). If the generating functional is real, the evolution of the 3-metric for one particular universe is given by

$$\left( \dot{\gamma}_{ij} - N_{i|j} - N_{j|i} \right) / N = 2\gamma^{-1/2} (2\gamma_{jk}\gamma_{il} - \gamma_{ij}\gamma_{kl}) \frac{\delta \mathcal{S}}{\delta \gamma_{kl}}, \quad (2.6a)$$

whereas the evolution equation for the scalar field is

$$\left( \dot{\phi} - N^i \phi_{,i} \right) / N = \gamma^{-1/2} \frac{\delta \mathcal{S}}{\delta \phi}. \quad (2.6b)$$

Here  $|$  denotes a covariant derivative with respect to the 3-metric  $\gamma_{ij}$ . The lapse and shift function appear neither in the HJ equation (2.1a) nor in the momentum constraint (2.1b). Hence in HJ theory, all gauge-dependent quantities appear only in the evolution equations, (2.6a) and (2.6b), for the metric and scalar field.

HJ methods can also be applied fruitfully to systems of perfect fluids [3], [37]. For example the Hamiltonian and momentum constraints for collisionless, pressureless dust [38] are given by:

$$\begin{aligned} 0 = \mathcal{H}(x) &= \gamma^{-1/2} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}(x)} [2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)] \\ &\quad + \sqrt{1 + \gamma^{ij}\chi_{,i}\chi_{,j}} \frac{\delta \mathcal{S}}{\delta \chi(x)} - \frac{1}{2}\gamma^{1/2}R + V_0, \end{aligned} \quad (2.7a)$$

$$0 = \mathcal{H}_i(x) = -2 \left( \gamma_{ik} \frac{\delta \mathcal{S}}{\delta \gamma_{kj}(x)} \right)_{,j} + \frac{\delta \mathcal{S}}{\delta \gamma_{kl}(x)} \gamma_{kl,i} + \frac{\delta \mathcal{S}}{\delta \chi(x)} \chi_{,i}. \quad (2.7b)$$

A cosmological constant term denoted by  $V_0$  has also been included. The dust field  $\chi$  describes, for example, cold-dark-matter particles. Its evolution equation is

$$\left( \dot{\chi} - N^i \chi_{,i} \right) / N = \sqrt{1 + \chi_{|k}\chi^{k|}}, \quad (2.8)$$

whereas that for the metric is given by eq.(2.6a). The 4-velocity  $U^\mu$  of the dust is the 4-gradient of the potential  $\chi$ ,

$$U^\mu = -g^{\mu\nu} \chi_{,\nu} , \quad (2.9)$$

where  $g^{\mu\nu}$  is the inverse of the 4-metric.

## A. Review of Long-Wavelength Solution

### 1. Scalar Field and Gravity

For fields where the wavelength is long compared to the Hubble radius, one may safely neglect second order spatial gradients (terms within square brackets) in the HJ equation (2.1a) for gravity interacting with a scalar field. The resulting equation has the trivial solution

$$\mathcal{S}^{(0)}[\gamma_{ij}(x), \phi(x)] = -2 \int d^3x \gamma^{1/2} H [\phi(x)] . \quad (2.10)$$

provided that the Hubble function  $H \equiv H(\phi)$ , a function of a single variable, satisfies the separated Hamilton-Jacobi equation of order zero [4]:

$$H^2 = \frac{2}{3} \left( \frac{\partial H}{\partial \phi} \right)^2 + \frac{V(\phi)}{3} . \quad (2.11)$$

Eq.(2.10) is also a solution of the momentum constraint (2.1b) because the volume element  $d^3x \gamma^{1/2}$  is invariant under reparametrizations of the spatial coordinates.  $H \equiv H(\phi)$  corresponds to the Hubble parameter in the long-wavelength limit. We will examine in detail the special case of inflation with an exponential potential [39]

$$V(\phi) = V_0 \exp \left( -\sqrt{\frac{2}{p}} \phi \right) , \quad (2.12a)$$

where  $p$  is a constant that describes the steepness of the potential. An exponential potential arises naturally in the induced gravity model (see, e.g., ref. [34]) as well as in extended inflation [40]. For this case, one can find the exact general solution [4] of eq.(2.11). In particular, the separated HJ equation of order zero has the attractor solution

$$H(\phi) = \left[ \frac{V_0}{3(1 - 1/(3p))} \right]^{1/2} \exp \left( -\frac{\phi}{\sqrt{2p}} \right) . \quad (\text{single scalar field}) \quad (2.12b)$$

It describes power-law inflation where the scale factors evolves as  $a(t) \propto t^p$ , with  $t$  being a synchronous time variable. (In order that the scalar field convert its energy into radiation and matter at the end of inflation, the scalar field should have a minimum in its potential. Hence, we interpret eq.(2.12a) as describing the asymptotic branch of the potential as  $\phi \rightarrow -\infty$ ; we will assume that microwave background fluctuations as well as fluctuations for galaxy formation are generated on this branch.) The separated Hamilton-Jacobi equation has been widely applied in the reconstruction of the inflaton potential from cosmological observations [41].

## 2. Dust Field and Gravity

For the case of a dust field, the long-wavelength theory is found by dropping  $R$  and  $\chi_{|i}\chi^{|i}$  in the HJ equation (2.7a). One can then attempt a solution analogous to eq.(2.10) except now  $H \equiv H(\chi)$  is a function of  $\chi$  satisfying

$$H^2 = -\frac{2}{3} \frac{\partial H}{\partial \chi} + H_0^2, \quad \text{where} \quad H_0^2 = \frac{V_0}{3}. \quad (2.13)$$

The general solution is

$$H(\chi) = H_0 \coth \left[ \frac{3H_0}{2} (\chi - \tilde{\chi}) \right], \quad (2.14a)$$

where  $\tilde{\chi}$  is a homogeneous constant [3]. If the vacuum energy density  $V_0$  is negligible, then one recovers the Hubble parameter for a matter dominated universe in the limit that the cosmological constant vanishes,  $H_0 \rightarrow 0$ :

$$H(\chi) = \frac{2}{3(\chi - \tilde{\chi})}. \quad (2.14b)$$

The scale factor,  $\gamma^{1/6}$ , defined to be the sixth root of the determinant of the 3-metric,

$$\gamma^{1/6} = d(x) \left( \frac{\partial H}{\partial \tilde{\chi}} \right)^{-1/3}, \quad (2.15)$$

can be found by taking the derivative with respect to the parameter  $\tilde{\chi}$  [3]. Here  $d(x)$  is an arbitrary function of the spatial coordinates. For dust with cosmological constant, we see that

$$\gamma^{1/6} \propto d(x) \left[ \sinh \left( \frac{3H_0}{2} (\chi - \tilde{\chi}) \right) \right]^{2/3}, \quad (2.16a)$$

whereas for pure dust, we recover the result

$$\gamma^{1/6} \propto d(x) (\chi - \tilde{\chi})^{2/3}. \quad (2.16b)$$

Eq.(2.16a) is plotted in Fig.(2).

## B. Field-Space Diagrams

The consequences of the long-wavelength approximation can be illuminated using *field-space diagrams* where one plots the metric variable versus the matter field.

### 1. Scalar Field and Gravity

For simplicity, we will assume that the shift function vanishes. The evolution equations (2.6a), (2.6b) in the long-wavelength limit then become

$$\dot{\gamma}_{ij}/N = 2H(\phi) \gamma_{ij}, \quad (2.17a)$$

$$\dot{\phi}/N = -2 \frac{\partial H}{\partial \phi}. \quad (2.17b)$$

These can be simplified by defining the field,

$$\alpha(t, x) = [\ln \gamma(t, x)] / 6, \quad (2.18a)$$

and then letting

$$\gamma_{ij}(t, x) = e^{2\alpha(t, x)} k_{ij}(x), \quad (2.18b)$$

where  $k_{ij}(x)$  is independent of time (this is the most interesting case) with  $\det(k)=1$ . Hence one obtains

$$\dot{\alpha}/N = H(\phi), \quad (2.19a)$$

$$\dot{\phi}/N = -2 \frac{\partial H}{\partial \phi}. \quad (2.19b)$$

We can integrate these equations by utilizing our freedom in choosing time. If we choose  $t = \phi$  to be the time variable, the lapse is defined through eq.(2.19b)

$$1/N = -2 \frac{\partial H}{\partial \phi}. \quad (2.20)$$

Eq.(2.19a) then becomes

$$\frac{d\alpha}{d\phi} = -\frac{1}{2} \frac{H(\phi)}{\frac{\partial H}{\partial \phi}}, \quad (2.21)$$

which involves only  $\alpha$  and the independent variable  $\phi$ . It may be integrated immediately leading to

$$\alpha(\phi, x) = \alpha_0(x) - \frac{1}{2} \int_0^\phi d\phi' \frac{H(\phi')}{\frac{\partial H}{\partial \phi'}} \quad (2.22)$$

where  $\alpha_0(x)$  is an arbitrary function of  $x$ . For the example of inflation with an exponential potential, eq.(2.12b), we obtain the trivial solution

$$\alpha(\phi, x) = \alpha_0(x) + \sqrt{\frac{p}{2}} \phi. \quad (2.23)$$

In general, the metric variable  $\alpha$  is inhomogeneous on a surface of uniform scalar field. For a single scalar field,  $\zeta/3$  is defined to be the metric fluctuation on a uniform  $\phi$  slice:

$$\zeta/3 \equiv \Delta\alpha(\phi) \equiv \alpha(\phi, x_2) - \alpha(\phi, x_1) = \alpha_0(x_2) - \alpha_0(x_1); \quad (2.24)$$

it is the difference of  $\alpha$  between two spatial points  $x_2$  and  $x_1$  on a time hypersurface of uniform  $\phi$  [14], [34], [4]. It is independent of time. For a single scalar field in the long-wavelength limit, this is true in general (see eq.(2.22)), and not just for the example of an exponential potential.

We plot the solutions eq.(2.23) in the field-space diagram Fig.(1). Each curve represents the evolution of the fields  $(\phi, \alpha)$  for a given spatial point. One may invert eq.(2.23) to obtain  $\phi$  as a function of  $\alpha$

$$\phi(\alpha, x) = -\sqrt{\frac{2}{p}}\alpha_0(x) + \sqrt{\frac{2}{p}}\alpha. \quad (2.25)$$

We interpret this inversion as choosing a time-hypersurface where  $\alpha$  is uniform. In this situation, the scalar field is inhomogeneous. The fluctuation in the scalar field between two spatial points  $x_2$  and  $x_1$  is given by

$$\Delta\phi(\alpha) \equiv \phi(\alpha, x_2) - \phi(\alpha, x_1) = -\sqrt{\frac{2}{p}} [\alpha_0(x_2) - \alpha_0(x_1)] = -\sqrt{\frac{2}{p}} \Delta\alpha(\phi). \quad (2.26)$$

It is related to the metric fluctuation through the negative of the slope of the  $\alpha$  versus  $\phi$  trajectories. The transformation from a surface of uniform  $\phi$  to one of uniform  $\alpha$  is simply visualized in the field-space diagram.

## 2. Dust Field and Gravity

A similar long-wavelength analysis can be repeated for the dust field. The evolution equations are

$$\dot{\alpha}/N = H(\chi), \quad (2.27a)$$

$$\dot{\chi}/N = 1. \quad (2.27b)$$

If  $\chi$  is taken to be the time hypersurface, then  $N = 1$ , and eq.(2.27a) may be integrated using the same method that was employed for a scalar field. More simply, one can just apply eq.(2.16a),

$$\alpha(\chi, x) = \alpha_0(x) + \frac{2}{3} \ln \sinh \left[ \frac{3H_0\chi}{2} \right]. \quad (2.28)$$

We have set  $\tilde{\chi} = 0$ , which may always be arranged by shifting  $\chi$ . We plot  $\alpha$  as a function of  $\chi$  in the field-space diagram, Fig.(2). Qualitatively it is the same as Fig.(1), except that here the trajectories are not straight lines.

HJ formalism is particularly useful for problems involving several fields. For example, in the long-wavelength limit, we have shown how to solve exactly the case of two fluids, a

dust field and a field describing blackbody radiation [37]. This solution describes adiabatic as well as isothermal perturbations. In this paper, we will consider only one matter field at a time. (Using a Taylor series expansion in synchronous time, Comer *et al* [42] have investigated various perfect fluids.)

These very simple considerations illustrate the role of time in general relativity. The HJ formalism appears to indicate that the most useful choices for the time hypersurface will be one of the matter fields or the metric variable  $\alpha$ , or some combination of the two. A hypersurface transformation amounts to slicing a field-space diagram in a particular direction. Such simple behavior is also manifested in the higher order solutions to the HJ equation for general relativity that we will consider in the next section.

### III. QUADRATIC CURVATURE APPROXIMATION FOR GRAVITY PLUS SCALAR FIELD

#### A. Factoring Out the Long-Wavelength Background

Before we attempt to solve the HJ equation (2.1a), we subtract out the long-wavelength background from the generating functional:

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{F} \ , \quad \mathcal{S}^{(0)} = -2 \int d^3x \gamma^{1/2} H(\phi) \ , \quad (3.1)$$

where  $H$  satisfies the separated Hamilton-Jacobi equation (2.11). The functional for fluctuations,  $\mathcal{F}$ , now satisfies

$$\begin{aligned} & -2 \frac{\partial H}{\partial \phi} \frac{\delta \mathcal{F}}{\delta \phi} + 2H \gamma_{ij} \frac{\delta \mathcal{F}}{\delta \gamma_{ij}} + \gamma^{-1/2} \frac{\delta \mathcal{F}}{\delta \gamma_{ij}(x)} \frac{\delta \mathcal{F}}{\delta \gamma_{kl}(x)} [2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)] \\ & + \frac{1}{2} \gamma^{-1/2} \left( \frac{\delta \mathcal{F}}{\delta \phi(x)} \right)^2 - \frac{1}{2} \gamma^{1/2} R + \frac{1}{2} \gamma^{1/2} \gamma^{ij} \phi_{,i} \phi_{,j} = 0 \ . \end{aligned} \quad (3.2)$$

Although superficially similar, this step differs in principle from the usual analysis of perturbations on an homogeneous background [33], [34], [18]. Here a long-wavelength background is allowed, which is closely related to what is done in stochastic inflation. The first term

$\mathcal{S}^{(0)}$  is explicitly invariant under reparametrizations of the spatial coordinates — gauge-invariance is manifestly maintained. Moreover, no explicit choice of the time parameter has been made.

The first line of eq.(3.2) may be simplified if one introduces a change of variables,  $(\gamma_{ij}, \phi) \rightarrow (f_{ij}, u)$ :

$$u = \int \frac{d\phi}{-2\frac{\partial H}{\partial \phi}} , \quad f_{ij} = \Omega^{-2}(u) \gamma_{ij} , \quad (3.3a)$$

where the conformal factor  $\Omega \equiv \Omega(u)$  is defined through

$$\frac{d \ln \Omega}{du} \equiv -2 \frac{\partial H}{\partial \phi} \frac{\partial \ln \Omega}{\partial \phi} = H . \quad (3.3b)$$

Functional derivatives with respect to the fields transform according to

$$\frac{\delta}{\delta \gamma_{ij}} = \Omega^{-2}(u) \frac{\delta}{\delta f_{ij}} \Big|_u , \quad \frac{\delta}{\delta \phi} = -\frac{1}{2} \frac{1}{\frac{\partial H}{\partial \phi}} \left( \frac{\delta}{\delta u} \Big|_{f_{ij}} - 2H f_{ij} \frac{\delta}{\delta f_{ij}} \Big|_u \right) . \quad (3.4)$$

In order to simplify the notation, we will henceforth suppress the symbols  $|_u$  and  $|_{f_{ij}}$  which denote the variables that are held constant during differentiation. Utilizing the conformal 3-metric  $f_{ij}$  instead the original 3-metric  $\gamma_{ij}$  is analogous to using comoving coordinates rather than physical coordinates in cosmological systems. At long-wavelengths, a surface of uniform  $u$  corresponds to comoving, synchronous gauge because  $N = 1$  in eq.(2.19b). Even if  $\phi$  (considered as a function of  $H$ ) oscillates,  $u$  is monotonic. However, when one considers short-wavelength terms associated with the functional  $\mathcal{F}$  for fluctuations, a surface of uniform  $u$  no longer represents a synchronous gauge.

The HJ equation reduces to

$$\frac{\delta \mathcal{F}}{\delta u} + \Omega^{-3}(u) f^{-1/2} \frac{\delta \mathcal{F}}{\delta f_{ij}} \frac{\delta \mathcal{F}}{\delta f_{kl}} [2f_{il}f_{jk} - f_{ij}f_{kl}] + \frac{\Omega^{-3}(u) f^{-1/2}}{8 \left( \frac{\partial H}{\partial \phi} \right)^2 (u)} \left[ \frac{\delta \mathcal{F}}{\delta u} - 2H(u) f_{ij} \frac{\delta \mathcal{F}}{\delta f_{ij}} \right]^2 = \frac{\delta \mathcal{S}^{(2)}}{\delta u} \quad (3.5)$$

where the functional  $\mathcal{S}^{(2)}$  is given by

$$\mathcal{S}^{(2)}[f_{ij}(x), u(x)] = \int d^3x f^{1/2} \left[ j(u) \tilde{R} + k(u) u^i u_{;i} \right] . \quad (3.6a)$$

From now on, a semi-colon will denote a covariant derivative with respect to the conformal 3-metric, e.g.,  $u^i \equiv f^{ij} u_{;j}$ . In addition,  $\tilde{R}$  is the Ricci scalar of  $f_{ij}$ . The  $u$ -dependent coefficients  $j$  and  $k$ ,

$$j(u) = \int_0^u \frac{\Omega(u')}{2} du' + F, \quad k(u) = H(u) \Omega(u) , \quad (3.6b)$$

where  $F$  is an arbitrary constant, were first derived in refs. [3], [5] in order that the spatial gradient terms appearing in eq.(3.2),

$$\frac{1}{2}\gamma^{1/2}R - \frac{1}{2}\gamma^{1/2}\gamma^{ij}\phi_{,i}\phi_{,j} = \frac{\delta\mathcal{S}^{(2)}}{\delta u} , \quad (3.7)$$

may be expressed as a functional derivative with respect to  $u(x)$  holding  $f_{ij}(x)$  fixed. The momentum constraint maintains the same form as before but it is now expressed in terms of the new variables  $(f_{ij}, u)$ .

$$\mathcal{H}_i(x) = -2 \left( f_{ik} \frac{\delta\mathcal{F}}{\delta f_{kj}} \right)_{,j} + \frac{\delta\mathcal{F}}{\delta f_{kl}} f_{kl,i} + \frac{\delta\mathcal{F}}{\delta u} u_{,i} = 0 . \quad (3.8)$$

## B. Integral form of HJ equation

It will prove useful to work with an integral form of the HJ equation. Before proceeding, we pause to consider a simple illustration from potential theory.

### 1. Potential Theory

The fundamental problem in potential theory is: given a force field  $g^i(u_k)$  which is a function of  $n$  variables  $u_k$ , what is the potential  $\Phi \equiv \Phi(u_k)$  (if it exists) whose gradient returns the force field:

$$\frac{\partial\Phi}{\partial u_i} = g^i(u_k) \quad ? \quad (3.9)$$

Not all force fields are derivable from a potential. Provided that the force field satisfies the integrability relation,

$$0 = \frac{\partial g^i}{\partial u_j} - \frac{\partial g^j}{\partial u_i} = \left[ \frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_i} \right] \Phi , \quad (3.10)$$

(i.e., it is curl-free), one may find a solution which is conveniently expressed using a line-integral

$$\Phi(u_k) = \int_C \sum_j dv_j g^j(v_l) . \quad (3.11)$$

If the two endpoints are fixed, all contours return the same answer. In practice, we will employ the simplest contour that one can imagine: a line connecting the origin to the observation point  $u_k$ . Using  $s$ ,  $0 \leq s \leq 1$ , to parameterize the contour, we define the variable  $v_l \equiv v_l(s, u_l)$ ,

$$v_l = s u_l \quad \text{with} \quad dv_l = ds u_l \quad (3.12)$$

so the line-integral may be rewritten as

$$\Phi(u_k) = \sum_{j=1}^n \int_0^1 \frac{ds}{s} v_j g^j(v_k) . \quad (3.13)$$

Similarly, in solving for the generating functional, we will employ a line-integral in superspace. The integrability condition for the HJ equation follows from the Poisson bracket of the constraints (Moncrief and Teitelboim [43])

$$\{\mathcal{H}(x^k), \mathcal{H}(x^{k'})\} = [\gamma^{ij}(x^k) \mathcal{H}_j(x^k) + \gamma^{ij}(x^{k'}) \mathcal{H}_j(x^{k'})] \frac{\partial}{\partial x^i} \delta^3(x^k - x^{k'}) . \quad (3.14)$$

In fact, each contour of the line-integral corresponds to a particular time-hypersurface choice. Provided that the generating functional is invariant under reparametrizations of the spatial coordinates, (e.g.,  $\mathcal{H}_j$  vanishes in eq.(3.14)), different time-hypersurface choices will lead to the same generating functional. Hypersurface invariance is closely related to gauge-invariance.

## 2. Line-Integral in Superspace

Following Parry, Salopek and Stewart [5], an integral form of the HJ eq.(3.5) may be constructed using a line-integral in superspace:

$$\begin{aligned} \mathcal{F}[f_{ij}(x), u(x)] + \int d^3x \int_0^1 \frac{ds}{s} v \Omega^{-3}(v) f^{-1/2} \frac{\delta \mathcal{F}}{\delta f_{ij}} \frac{\delta \mathcal{F}}{\delta f_{kl}} [2f_{il}f_{jk} - f_{ij}f_{kl}] + \\ \int d^3x \int_0^1 \frac{ds}{s} \frac{v \Omega^{-3}(v) f^{-1/2}}{8 \left( \frac{\partial H}{\partial \phi}(v) \right)^2} \left[ \frac{\delta \mathcal{F}}{\delta v} - 2H(v) f_{ij} \frac{\delta \mathcal{F}}{\delta f_{ij}} \right]^2 = \mathcal{S}^{(2)}[f_{ij}(x), u(x)] \end{aligned} \quad (3.15)$$

In analogy with the example from potential theory, we replace the index  $j$  with the spatial coordinate  $x$  and the finite sum  $\sum_j$  with the integral  $\int d^3x$ . The integrating parameter is again denoted by  $s$ , and  $v(x) = su(x)$  represents a straight line in the superspace of the *scalar field*; however, at each spatial point,  $f_{ij}(x)$  is held fixed in the line-integral. As a result, one may safely add an arbitrary functional of  $f_{ij}$  to the right-hand side. No approximations have been made thus far; eq.(3.15) is exact.

### C. ANSATZ

For a spatial gradient expansion of  $\mathcal{S}$  we have already shown how to set up a recursion relation [5] which gives higher order terms from the previous orders. Explicit solutions were given which were accurate to fourth order in spatial gradients. However, we now wish to investigate higher order terms. Finding an explicit and exact expression which is valid to all orders appears to be extremely difficult. Instead we will examine an infinite subset consisting of all terms quadratic in the Ricci curvature, but containing any number of spatial gradients. Halliwell and Hawking [33] used a similar approximation, although our expansion will manifestly maintain hypersurface and gauge-invariance. We hence make an Ansatz of the form,

$$\mathcal{F} = \mathcal{S}^{(2)} + \mathcal{Q}, \quad \text{with} \quad (3.16)$$

$$\mathcal{Q} = \int d^3x f^{1/2} \left[ \tilde{R} \hat{S}(u, \tilde{D}^2) \tilde{R} + \tilde{R}^{ij} \hat{T}(u, \tilde{D}^2) \tilde{R}_{ij} - \frac{3}{8} \tilde{R} \hat{T}(u, \tilde{D}^2) \tilde{R} \right], \quad (3.17)$$

where  $\hat{S}(u, \tilde{D}^2)$  and  $\hat{T}(u, \tilde{D}^2)$  are differential operators which are also functions of  $u$ .  $\tilde{D}^2$  is the Laplacian operator with respect to the conformal 3-metric, e.g.,

$$\tilde{D}^2 \tilde{R} \equiv \tilde{D}^{;i} \tilde{D}_{;i} \tilde{R} \equiv \tilde{R}^{;i}_{;i} = f^{-1/2} \left( f^{1/2} f^{ij} \tilde{R}_{,j} \right)_{,i}. \quad (3.18)$$

We refer to  $\mathcal{Q}$  as the quadratic functional. We interpret the operator  $\hat{S}$  for scalar perturbations to be a Taylor series of the form

$$\hat{S}(u, \tilde{D}^2) = \sum_n S_n(u) (\tilde{D}^2)^n, \quad (3.19)$$

and similarly for the operator  $\hat{T}$  describing tensor fluctuations. Note that  $\hat{T}$  is sandwiched between two Ricci tensors,  $\tilde{R}_{ij}$ , as well as two Ricci scalars  $\tilde{R}$ . The full Riemann tensor  $\tilde{R}_{ijkl}$  does not appear in the Ansatz because for three spatial dimensions because it may be written in terms of the Ricci tensor (see, e.g., [44]). Although the first and third terms in eq.(3.17) may be combined into a single one, the present form simplifies the final evolution equations for  $\hat{S}$  and  $\hat{T}$  (see eqs.(3.27a,b)).

### 1. Order of Perturbation

We now define the rules which determine the order in perturbation of various terms. By first order, we refer to terms such as  $\tilde{R}$  or  $\tilde{D}^2 u$ , or  $\tilde{D}^2 \tilde{R}$  or  $\tilde{D}^4 u$ , which vanish if the fields are homogeneous; they may contain any number of spatial derivatives. Quadratic terms are a product of two linear terms. Some examples are  $u_{;i} u^{;i}$ ,  $\tilde{D}^2 u \tilde{D}^4 \tilde{R}$ . It should be clear how to determine other cases.

Other quadratic terms such as  $(\tilde{D}^2 u)^2$  and  $\tilde{R} \tilde{D}^2 u$ , could be included in the Ansatz (3.17) as well. However for the case of a single scalar field as well as that for a single dust field, it may be shown that they actually vanish. This is one of advantages of utilizing the conformal transformation (3.3a,b). (This simplification does not occur for multiple matter fields, and analogous terms would have to be included.)

All that is necessary now is to compute some functional derivatives, and substitute into the integrated HJ equation (3.15). It is useful to note that for a small variation of the conformal 3-metric  $\delta f_{ij}$  the corresponding change in the Ricci tensor is

$$\delta \tilde{R}_{ij} = \frac{1}{2} f^{kl} [\delta f_{li;jk} + \delta f_{lj;ik} - \delta f_{ij;lk} - \delta f_{lk;ij}]. \quad (3.20)$$

In the integral form of the HJ equation, integration by parts is permitted which simplifies the analysis considerably. (However, in its differential form (2.1a) one cannot simply discard

total spatial derivatives.) In addition, all cubic terms are neglected. For example from eqs.(3.6a,b) we find that

$$\frac{\delta \mathcal{S}^{(2)}}{\delta u} - 2H f_{ij} \frac{\delta \mathcal{S}^{(2)}}{\delta f_{ij}} = f^{1/2} \left( \frac{\Omega}{2} - H j \right) \tilde{R}, \quad (3.21)$$

where a term proportional to  $u_{;i} u^i$  has been dropped, otherwise if (3.21) were squared, this term's contribution to the integral HJ equation (3.15) would be cubic and higher which we are not considering here. Other terms that appear are

$$\begin{aligned} \int d^3x \int_0^1 \frac{ds}{s} v \Omega^{-3}(v) f^{-1/2} \frac{\delta \mathcal{S}^{(2)}}{\delta f_{ij}} \frac{\delta \mathcal{S}^{(2)}}{\delta f_{kl}} [2f_{il} f_{jk} - f_{ij} f_{kl}] = \\ \int d^3x \int_0^1 \frac{ds}{s} f^{1/2} 2v \Omega^{-3}(v) j^2(v) \left( \tilde{R}^{ij} \tilde{R}_{ij} - \frac{3}{8} \tilde{R}^2 \right). \end{aligned} \quad (3.22)$$

The order of some operators may be changed if the commutator is an undesired higher order term, e.g.,

$$\int f^{1/2} \tilde{R} \tilde{D}^2 \hat{T} \tilde{R} = \int f^{1/2} \tilde{R} \hat{T} \tilde{D}^2 \tilde{R}. \quad (3.23)$$

Hence,  $\tilde{D}^2$  effectively commutes with any function of  $u$ . For a similar reason, one may permute the order of spatial differentiation with impunity,  $\tilde{D}_{;i} \tilde{D}_{;j} \sim \tilde{D}_{;j} \tilde{D}_{;i}$ , because the additional Riemann tensor term would be of higher order. As a result, we find

$$\begin{aligned} \int d^3x \int_0^1 \frac{ds}{s} v \frac{\Omega^{-3}(v) f^{-1/2}}{8 \left( \frac{\partial H}{\partial \phi}(v) \right)^2} \left[ \frac{\delta \mathcal{F}}{\delta v} - 2H f_{ij} \frac{\delta \mathcal{F}}{\delta f_{ij}} \right]^2 = \int d^3x f^{1/2} \tilde{R} \int_0^u dv \left\{ \frac{1}{8 \Omega^3(v) \left( \frac{\partial H}{\partial \phi}(v) \right)^2} \right. \\ \left. \left[ 64 H^2(v) \hat{S}^2 \tilde{D}^4 + 16 H(v) \left( \frac{\Omega(v)}{2} - H(v) j(v) \right) \hat{S} \tilde{D}^2 + \left( \frac{\Omega(v)}{2} - H(v) j(v) \right)^2 \right] \right\} \tilde{R}, \end{aligned} \quad (3.24)$$

and that

$$\begin{aligned} \int d^3x \int_0^1 \frac{ds}{s} v \Omega^{-3} f^{-1/2} \frac{\delta \mathcal{F}}{\delta f_{ij}} \frac{\delta \mathcal{F}}{\delta f_{kl}} [2f_{il} f_{jk} - f_{ij} f_{kl}] = \\ \int d^3x f^{1/2} \left\{ \tilde{R}^{ij} \int_0^u dv \Omega^{-3} \left[ 2j^2 + 4j \hat{T} \tilde{D}^2 + 2 \hat{T}^2 \tilde{D}^4 \right] \tilde{R}_{ij} \right. \\ \left. - \frac{3}{8} \tilde{R} \int_0^u dv \Omega^{-3} \left[ 2j^2 + 4j \hat{T} \tilde{D}^2 + 2 \hat{T}^2 \tilde{D}^4 \right] \tilde{R} \right\}. \end{aligned} \quad (3.25)$$

Collecting terms, the integral form of the HJ equation (3.15) gives:

$$0 = \int d^3x f^{1/2} \left\{ \tilde{R} \left[ \hat{S} + \int_0^u dv \frac{1}{8\Omega^3(v) \left( \frac{\partial H}{\partial \phi}(v) \right)^2} \left( \frac{\Omega(v)}{2} - H(v)j(v) + 8H(v)\hat{S}\widetilde{D}^2 \right)^2 \right] \tilde{R} + \right. \\ \left. \tilde{R}^{ij} \left[ \hat{T} + \int_0^u dv \frac{2}{\Omega^3(v)} (j(v) + \hat{T}\widetilde{D}^2)^2 \right] \tilde{R}_{ij} - \frac{3}{8} \tilde{R} \left[ \hat{T} + \int_0^u dv \frac{2}{\Omega^3(v)} (j(v) + \hat{T}\widetilde{D}^2)^2 \right] \tilde{R} \right\}. \quad (3.26)$$

Each coefficient must vanish separately, leading to a pair of uncoupled integral equations.

Taking the derivative with respect to  $u$  for each, one obtains the differential equations

$$0 = \frac{\partial \hat{S}}{\partial u} + \frac{1}{8\Omega^3 \left( \frac{\partial H}{\partial \phi} \right)^2} \left( \frac{\Omega}{2} - Hj + 8H\hat{S}\widetilde{D}^2 \right)^2, \quad (3.27a)$$

$$0 = \frac{\partial \hat{T}}{\partial u} + \frac{2}{\Omega^3} (j + \hat{T}\widetilde{D}^2)^2. \quad (3.27b)$$

Since one may add an arbitrary functional of  $f_{ij}$  to the integral HJ eq.(3.15), these first order, ordinary differential equations are equivalent to the pair of integral equations arising from eq.(3.26). These nonlinear differential equations are of the Riccati type. They represent a tremendous simplification over the original HJ equation (2.1a).

#### D. SOLVING RICCATI EQUATIONS

As is well known, the Riccati equations, (3.27a) and (3.27b), may be reduced to linear ordinary differential equations. To this aim, we define the Riccati transformation

$$w \equiv w(u, \widetilde{D}^2), \quad y \equiv y(u, \widetilde{D}^2), \quad (3.28)$$

through

$$\hat{S} = \frac{\Omega^3 \left( \frac{\partial H}{\partial \phi} \right)^2}{8H^2 \widetilde{D}^4} \frac{1}{w} \frac{\partial w}{\partial u} + \frac{2Hj - \Omega}{16H \widetilde{D}^2}, \quad (3.29)$$

$$\hat{T} = \frac{\Omega^3}{2\widetilde{D}^4} \frac{1}{y} \frac{\partial y}{\partial u} - \widetilde{D}^{-2}j, \quad (3.30)$$

which lead to the desired result:

$$0 = \frac{\partial^2 w}{\partial u^2} + \left\{ 3H(u) + 2 \frac{\partial}{\partial u} \left[ \ln \left( \frac{1}{H} \frac{\partial H}{\partial \phi} \right) \right] \right\} \frac{\partial w}{\partial u} - \Omega^{-2}(u) \widetilde{D}^2 w, \quad (\text{scalar perturbations}) \quad (3.31)$$

$$0 = \frac{\partial^2 y}{\partial u^2} + 3H(u) \frac{\partial y}{\partial u} - \Omega^{-2}(u) \widetilde{D}^2 y. \quad (\text{tensor perturbations}) \quad (3.32)$$

Equation (3.31) describes the evolution of scalar perturbations (density perturbations) of the metric, whereas eq.(3.32) describes tensor perturbations (gravitational waves). We emphasize that no time choice has been made in deriving these basic equations, although the scalar field parametrizes the evolution because  $u \equiv u(\phi)$  is the independent variable. These are the fundamental equations of cosmological perturbations that we will utilize in this paper.

If we set,

$$w = -\frac{1}{2} \frac{H}{\frac{\partial H}{\partial \phi}} z , \quad (3.33a)$$

we obtain the equation for scalar perturbations that was derived by Mukhanov *et al* [18]

$$0 = \frac{\partial^2 z}{\partial u^2} + 3H(u) \frac{\partial z}{\partial u} + (m_{eff}^2 - \Omega^{-2} \widetilde{D}^2) z , \quad (3.33b)$$

where the effective mass  $m_{eff}$  is given by

$$m_{eff}^2 = \frac{\partial^2 V}{\partial \phi^2} + 2 \frac{\partial H}{\partial u} \left[ 3 - \frac{1}{H^2} \frac{\partial H}{\partial u} + 2 \frac{\partial^2 \phi}{\partial u^2} \middle/ \left( H \frac{\partial \phi}{\partial u} \right) \right] . \quad (3.33c)$$

In deriving these equations, it is useful to note several relationships between the scalar field  $\phi$  and the new variable  $u$  which follow from eqs.(3.3a, b) and (2.11):

$$\frac{d\phi}{du} = -2 \frac{\partial H}{\partial \phi} , \quad (3.34a)$$

$$\frac{d^2 \phi}{du^2} = -3H \frac{d\phi}{du} - \frac{\partial V}{\partial \phi} , \quad (3.34b)$$

$$\frac{d^3 \phi}{du^3} \middle/ \frac{d\phi}{du} = -3 \frac{dH}{du} - \frac{\partial^2 V}{\partial \phi^2} - 3H \frac{d^2 \phi}{du^2} \middle/ \frac{d\phi}{du} . \quad (3.34c)$$

Eq.(3.34b) is the well-known evolution equation for a long-wavelength scalar field. It is identical to the evolution equation in synchronous gauge ( $N = 1$ ) describing a homogeneous Friedman universe.

### 1. Exact Solution

Many exact solutions have been found for power-law inflation (see, e.g., eq.(2.12b)). Abbott and Wise [45] have shown that the evolution equation (3.32) for tensor perturbations

can be solved exactly. In addition, Lyth and E. Stewart [46] have shown that the same solution may be applied to the scalar perturbation eq.(3.33b). For example, if we use  $H(\phi)$  defined in eq.(2.12b), we find that

$$\frac{1}{H} \frac{\partial H}{\partial \phi} = -\frac{1}{\sqrt{2p}} \quad (3.35)$$

is a constant, and the equation for scalar perturbations eq.(3.31) becomes

$$0 = \frac{\partial^2 w}{\partial u^2} + \frac{3p}{u} \frac{\partial w}{\partial u} - u^{-2p} \widetilde{D}^2 w. \quad (\text{scalar perturbations}) \quad (3.36)$$

Here we have also used the fact that for power-law inflation, the Hubble parameter and  $\Omega$  are given by eqs.(3.3a,b)

$$H(u) = \frac{p}{u}, \quad \text{and} \quad \Omega(u) = u^p. \quad (3.37)$$

Eq.(3.36) is identical to that describing a *massless* scalar field in a Friedman universe. This is quite surprising since one would have naively expected that the mass term  $\partial^2 V / \partial \phi^2$  in eq.(3.33c) would lead to damping of the fluctuations with wavelengths larger than the Hubble radius but evidently the additional terms cancel this effect. In fact, the equation (3.36) for scalar perturbations is identical to the tensor evolution eq.(3.32). The solution for both of these equations can be expressed in terms of a Hankel function of the first kind:

$$y(u, \widetilde{D}^2) = z(u, \widetilde{D}^2) = \sqrt{\frac{2}{p}} w(u, \widetilde{D}^2) = \sqrt{\frac{\pi u^{(1-3p)}}{4|p-1|}} H_{\nu}^{(1)} \left[ \frac{\sqrt{-\widetilde{D}^2} u^{1-p}}{(1-p)} \right], \quad \text{with} \quad \nu = \frac{(3p-1)}{2(p-1)}. \quad (3.38)$$

The normalization of this solution is irrelevant since a logarithmic derivative enters in the Riccati transformation (3.29), (3.30). However, in order to agree with the conventions of Birrell and Davies [47] we have assumed that

$$\Omega^3(u) \left( y^* \frac{\partial y}{\partial u} - y \frac{\partial y^*}{\partial u} \right) = i = \Omega^3(u) \left( z^* \frac{\partial z}{\partial u} - z \frac{\partial z^*}{\partial u} \right). \quad (3.39)$$

Note that the argument for the Hankel function is negative, and it increases to zero as  $u$  varies from 0 to  $\infty$ . For an inflationary epoch where  $p > 1$ , it describes a positive frequency

solution as  $u \rightarrow 0$ , e.g., when the wavelength of a particular mode is far within the Hubble radius,

$$y = z \sim \frac{(-\tilde{D}^2)^{-1/4}}{\sqrt{2\Omega^2(u)}} \exp\left(i\sqrt{-\tilde{D}^2} \int \frac{du}{\Omega(u)}\right) \quad (u \rightarrow 0). \quad (3.40)$$

As a result, eq.(3.38) corresponds to the ground state wavefunctional at wavelengths shorter than the Hubble radius (i.e., the Bunch-Davies vacuum [25]; see also ref. [34]). However, when the wavelength exceeds the Hubble radius, the state is no longer in the ground state. (There is also an analogous solution to eq.(3.36) which involves a Hankel function of the second kind but it describes a negative frequency solution at short wavelengths — it does not describe a system which is initially in the ground state.)

#### IV. QUADRATIC CURVATURE APPROXIMATION FOR GRAVITY PLUS DUST FIELD

The quadratic curvature approximation used in Sec. III may be applied to any field that is derived from an action principle. We derive the scalar and tensor equations corresponding to a dust field. Unfortunately one cannot define a ground state for the dust field as was the case for a scalar field. Hence the conditions at the beginning of the dust-dominated era were generated at earlier times, such as during the scalar field dominated epoch of inflation.

We follow the treatment given for a scalar field. In analogy to eq.(3.1), we express the generating functional  $\mathcal{S} \equiv \mathcal{S}[\gamma_{ij}(x), \chi(x)]$  for dust and gravity as the sum of a long-wavelength part  $\mathcal{S}^{(0)}$  and a fluctuation part  $\mathcal{F}$ . Here, however, the Hubble function  $H \equiv H(\chi)$  is given by eq.(2.13). The HJ equation (2.7a) for dust then becomes

$$\begin{aligned} \frac{\delta \mathcal{F}}{\delta \chi} + 2H\gamma_{ij}\frac{\delta \mathcal{F}}{\delta \gamma_{ij}} + \gamma^{-1/2}\frac{\delta \mathcal{F}}{\delta \gamma_{ij}(x)}\frac{\delta \mathcal{F}}{\delta \gamma_{kl}(x)}[2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)] \\ - \frac{1}{2}\gamma^{1/2}R + \left(\sqrt{1 + \chi^{|i}\chi_{|i}} - 1\right)\left(\frac{\delta \mathcal{F}}{\delta \chi} - 2\gamma^{1/2}\frac{\partial H}{\partial \chi}\right). \end{aligned} \quad (4.1)$$

Since we cannot solve this equation exactly, we will retain only those terms which are at most quadratic in perturbation:

$$\left(\sqrt{1 + \chi^{|i} \chi_{|i}} - 1\right) \left(\frac{\delta \mathcal{F}}{\delta \chi} - 2\gamma^{1/2} \frac{\partial H}{\partial \chi}\right) \sim \frac{1}{2} \chi^{|i} \chi_{|i} \left(\frac{\delta \mathcal{F}}{\delta \chi} - 2\gamma^{1/2} \frac{\partial H}{\partial \chi}\right) \sim -\gamma^{1/2} \chi^{|i} \chi_{|i} \frac{\partial H}{\partial \chi}. \quad (4.2)$$

We will also employ a conformal transformation of the 3-metric  $\gamma_{ij} \rightarrow f_{ij}$

$$f_{ij} = \Omega^{-2}(\chi) \gamma_{ij}, \quad (4.3)$$

where  $\Omega \equiv \Omega(\chi)$  is defined through

$$\frac{d \ln \Omega}{d \chi} = H(\chi). \quad (4.4)$$

(For example, if the vacuum energy density  $V_0$  vanishes in eq.(2.13), the Hubble function  $H$  and the conformal factor can be written as

$$H = \frac{2}{3\chi}, \quad \Omega = \chi^{2/3}, \quad (\text{with } \tilde{\chi} = 0). \quad (4.5)$$

Hence, the HJ equation for dust becomes

$$\left. \frac{\delta \mathcal{F}}{\delta \chi} \right|_{f_{ij}} + \Omega^{-3}(\chi) f^{-1/2} \frac{\delta \mathcal{F}}{\delta f_{ij}} \frac{\delta \mathcal{F}}{\delta f_{kl}} [2f_{il} f_{jk} - f_{ij} f_{kl}] = \left. \frac{\delta \mathcal{S}^{(2)}}{\delta \chi} \right|_{f_{ij}} \quad (4.6)$$

where the functional  $\mathcal{S}^{(2)}$  is given by

$$\mathcal{S}^{(2)}[f_{ij}(x), \chi(x)] = \int d^3x f^{1/2} [j(\chi) \tilde{R} + k(\chi) \chi^{;i} \chi_{;i}] ; \quad (4.7a)$$

once again, a covariant derivative with respect to  $f_{ij}$  is denoted using a semi-colon. The  $\chi$ -dependent coefficients  $j$  and  $k$  are [3], [5],

$$j(\chi) = \int_0^\chi \frac{\Omega(\chi')}{2} d\chi' + F, \quad k(\chi) = H(\chi) \Omega(\chi), \quad (4.7b)$$

where  $F$  is a constant, so that

$$\frac{1}{2} \gamma^{1/2} R + \gamma^{1/2} \frac{\partial H}{\partial \chi} \chi^{|i} \chi_{|i} = \left. \frac{\delta \mathcal{S}^{(2)}}{\delta \chi} \right|_{f_{ij}} \quad (4.8)$$

may be expressed as a functional derivative with respect to  $\chi(x)$  holding  $f_{ij}$  fixed. Using a line-integral in superspace, we can construct the integral form of this equation analogous to eq.(3.15):

$$\mathcal{F}[f_{ij}(x), \chi(x)] + \int d^3x \int_0^1 \frac{ds}{s} v \Omega^{-3}(v) f^{-1/2} \frac{\delta \mathcal{F}}{\delta f_{ij}} \frac{\delta \mathcal{F}}{\delta f_{kl}} [2f_{il}f_{jk} - f_{ij}f_{kl}] = \mathcal{S}^{(2)}[f_{ij}(x), \chi(x)], \quad (4.9)$$

where  $v(x) = s\chi(x)$  represents a straight line in superspace. This equation has a similar but simpler form than the case of a scalar field. We make the analogous quadratic Ansatz as in eq.(3.17), and we find that the evolution equations for the scalar and tensor operators,  $\hat{S} \equiv \hat{S}(\chi, \widetilde{D}^2)$  and  $\hat{T} \equiv \hat{T}(\chi, \widetilde{D}^2)$ , are

$$0 = \frac{\partial \hat{S}}{\partial \chi}, \quad (4.10a)$$

$$0 = \frac{\partial \hat{T}}{\partial \chi} + \frac{2}{\Omega^3(\chi)} \left( j(\chi) + \hat{T} \widetilde{D}^2 \right)^2. \quad (4.10b)$$

The solution to the scalar perturbation equation (4.10a) is trivial:

$$\hat{S} \equiv \hat{S}(\widetilde{D}^2) \quad (\text{scalar perturbations}) \quad (4.11)$$

is an arbitrary functional of the conformal Laplacian operator. It is independent of the dust field  $\chi$ . Since dust never oscillates like a scalar field, one cannot identify a ground state for the dust field. The tensor perturbation equation (4.10b) may be solved using the same technique that was employed for a scalar field. We first define  $y$ ,

$$\hat{T} = \frac{\Omega^3}{2\widetilde{D}^4} \frac{1}{y} \frac{\partial y}{\partial \chi} - \widetilde{D}^{-2} j(\chi), \quad (4.12)$$

which leads to a linear equation,

$$0 = \frac{\partial^2 y}{\partial \chi^2} + 3H(\chi) \frac{\partial y}{\partial \chi} - \Omega^{-2}(\chi) \widetilde{D}^2 y, \quad (\text{tensor perturbations}) \quad (4.13)$$

describing the evolution of the graviton in a universe with dust.

If the cosmological constant vanishes, we obtain the exact solution:

$$y = A\chi^{-1/2} J_{\frac{3}{2}} \left[ 3\sqrt{\widetilde{D}^2} \chi^{1/3} \right] + B\chi^{-1/2} J_{-\frac{3}{2}} \left[ 3\sqrt{\widetilde{D}^2} \chi^{1/3} \right]. \quad (4.14)$$

The choice of the coefficients  $A$  and  $B$  will be given in the next section. In fact, the forms for  $\hat{S}$  and  $\hat{T}$  at the beginning of the matter-dominated epoch are determined by a preceding period of inflation with a scalar field.

## V. LARGE ANGLE MICROWAVE BACKGROUND FLUCTUATIONS AND GALAXY CORRELATIONS

We first show how to compute large angle microwave background anisotropies arising from both the scalar and tensor fluctuations of inflation. In principle this is not difficult since it essentially amounts to expanding in spherical harmonics. We first interpret the semi-classical wavefunctional that was computed in the previous sections.

By maintaining gauge and hypersurface invariance in the dynamical analysis, we have not made any extraneous assumption which would typically complicate the results. Consequently, the final equations for a scalar field, (3.31), (3.32) and for a dust field, (4.10a), (4.13) are of a very simple form. However, many measurement processes actually choose a specific time-hypersurface. A simple example arises in special relativity: when measuring the lifetime of a particle, one typically chooses the rest frame of that particle. In an application of high interest to cosmology, Sachs and Wolfe [48] derived the large angle microwave background anisotropy by utilizing comoving, synchronous gauge (uniform  $\chi$  slice).

### A. Interpretation of Semi-Classical Wavefunctional

In earlier papers [7] – [9], it was shown that the gradient expansion could be used to compute nonlinear effects in cosmology. In particular, we computed higher order corrections to the Zel’dovich approximation. However in computing microwave background anisotropies, it is sufficient to consider a linear approximation; we will now consider only a small deviation  $h_{ij}$  of the conformal 3-metric  $f_{ij}$  from flat space  $\delta_{ij}$ , where the probability functional reaches its maximum:

$$f_{ij}(u, x) = \delta_{ij} + h_{ij}(u, x) \quad \text{with} \quad h_{ij}(u, x) = \sum_{a=1}^6 \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} h_a(u, k) E_{ij}^{(a)}(k). \quad (5.1)$$

We have expressed  $h_{ij}$  as a sum of plane waves with comoving wavenumber  $\vec{k}$ . Furthermore, we also expanded it in a complete basis of 6 symmetric, polarization matrices  $E_{ij}^{(a)}(k)$ . For example, if the comoving wavenumber is aligned with the z-axis, we choose them to be:

$$E_{ij}^{(1)} = \text{Diag}[1, -1, 0], \quad (\text{tensor}) \quad (5.2a)$$

$$E_{12}^{(2)} = E_{21}^{(2)} = 1, \quad (\text{all other components vanish}) \quad (\text{tensor}) \quad (5.2b)$$

$$E_{ij}^{(3)} = \text{Diag}[1, 1, 0], \quad (\text{scalar}) \quad (5.2c)$$

$$E_{ij}^{(4)} = \text{Diag}[0, 0, \sqrt{2}], \quad (\text{gauge}) \quad (5.2d)$$

$$E_{13}^{(5)} = E_{31}^{(5)} = 1 = E_{23}^{(6)} = E_{32}^{(6)}. \quad (\text{all other components vanish}) \quad (\text{gauge}) \quad (5.2e)$$

The first two are traceless and divergenceless; they correspond to tensor perturbations. The third describes scalar perturbations, whereas the remainder are gauge modes. The matrices have been normalized so that

$$E_{ij}^{(a)} E_{ij}^{(b)} = 2\delta^{ab}. \quad (5.3)$$

In addition, one must respect the reality condition:

$$h_a(k) = h_a^*(-k). \quad (5.4)$$

To linear order, the Ricci tensor for the conformal metric may be computed using eq.(3.20) with  $f_{ij} = \delta_{ij}$  and  $\delta f_{ij} = h_{ij}$  to give

$$\tilde{R}_{ij} = \frac{1}{2} \left( h_{li,j}{}^l + h_{lj,i}{}^l - h_{ij,l}{}^l - h_{,ij} \right). \quad (5.5)$$

The probability functional arising from inflation is then given by the square of the wave-functional, eq.(2.4):

$$|\Psi|^2[\gamma_{ij}(x), \phi(x)] = \exp - \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ |\beta_1(k)|^2 + |\beta_2(k)|^2 + |\beta_3(k)|^2 \right], \quad (5.6a)$$

where

$$h_1(k) = \frac{1}{\sqrt{2k^4 T_I(u, -k^2)}} \beta_1(k), \quad h_2(k) = \frac{1}{\sqrt{2k^4 T_I(u, -k^2)}} \beta_2(k), \quad h_3(k) = \frac{1}{\sqrt{16k^4 S_I(u, -k^2)}} \beta_3(k). \quad (5.6b)$$

Here  $\tilde{D}^2$  has been replaced by  $-k^2$ . In Fourier space,  $\hat{S}$  and  $\hat{T}$  are complex numbers which we expand into real and imaginary parts:

$$\hat{S}(u, -k^2) = S_R(u, -k^2) + iS_I(u, -k^2), \quad \text{and} \quad \hat{T} = T_R(u, -k^2) + iT_I(u, -k^2), \quad (5.7)$$

with

$$S_I(u, -k^2) = \frac{1}{16k^4} \frac{\left(\frac{\partial H}{\partial \phi}\right)^2}{H^2} \frac{1}{|z|^2}, \quad T_I(u, -k^2) = \frac{1}{4k^4} \frac{1}{|y|^2}, \quad (5.8)$$

which follows from the Riccati transformation equations (3.29), (3.30) and the normalization conditions (3.39); (we have assumed that  $u$  is uniform (see below) and  $j(u)$  is real). Because the wavefunctional is invariant under reparameterizations of the spatial coordinates, the gauge modes  $h_4(k)$ ,  $h_5(k)$  and  $h_6(k)$  are absent in eq.(5.6a). Hence they are unrestricted, and they may assume arbitrary values consistent with the reality condition (5.4). The three  $\beta$ 's are Gaussian random fields which satisfy the following:

$$< \beta_a(\vec{k}) \beta_b^*(\vec{k}) > = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \delta_{ab}, \quad \beta_r(\vec{k}) = \beta_r^*(-\vec{k}), \quad a, b = 1, 2, 3. \quad (5.9)$$

The polarization matrices have been chosen so that the probability functional (5.6a) is diagonal in  $h_1(k)$ ,  $h_2(k)$  and  $h_3(k)$ . (Using a canonical transformation in conjunction with HJ theory, Langlois [49] has derived a reduced phase space Hamiltonian; our approach differs from his in that we perform a phase space reduction after finding a solution to the HJ equation.)

### 1. Long-wavelength Fields from Power-law Inflation

*Long-wavelength fields* are measurable and their evolution was discussed in Sec. II.A. Recall that we defined  $\zeta/3$  to be the fluctuation in  $\alpha = (\ln \gamma)/6$  on a comoving slice, eq.(2.24). By choosing  $u \equiv u(\phi)$  to be uniform in eq.(5.6a), we see that for small deviations from flat space that zeta is related to  $h_3(k)$  through

$$\zeta(x) = \frac{3}{2} \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} h_3(k). \quad (5.10)$$

Since the gauge modes are arbitrary, we have chosen  $h_4 = h_3/\sqrt{2}$  in order that

$$h_3(u, k) E_{ij}^{(3)}(k) + h_4(u, k) E_{ij}^{(4)}(k) = h_3(u, k) \delta_{ij} \quad (5.11)$$

is proportional to the identity matrix. It has become conventional to define the power spectrum through

$$\mathcal{P}_\zeta(k) \equiv \frac{k^3}{2\pi^2} \int d^3x e^{-i\vec{k}\cdot\vec{x}} \langle \zeta(x)\zeta(0) \rangle, \quad (5.12)$$

which leads to

$$\mathcal{P}_\zeta(k) \equiv \frac{k^3}{2\pi^2} \frac{9H^2}{4\left(\frac{\partial H}{\partial \phi}\right)^2} |z|^2 = \frac{k^3}{2\pi^2} 9|w|^2. \quad (5.13)$$

Loosely speaking,  $[k^3|w|^2/(2\pi^2)]^{1/2}$  can be interpreted as the metric fluctuation,  $\Delta\alpha(\phi) = \zeta/3$ , on a uniform  $\phi$  slice (see eq.(2.24)), whereas  $[k^3|z|^2/(2\pi^2)]^{1/2}$  is the fluctuation in the scalar field,  $\Delta\phi(\alpha)$ , on a uniform  $\alpha$  (uniform curvature) slice (see eq.(2.26)). They are related through the hypersurface transformation eq.(3.33a); see Sec.II.B and Fig.(1). (In fact, Hwang [50] has used uniform curvature slices to provide an elegant derivation of the scalar perturbation equation (3.33b).) In DeSitter space where the scale factor varies exponentially in synchronous time,  $\Omega(u) = e^{Hu}$ , where  $H$  is a constant, it is useful to note that that at long-wavelengths

$$\left[ \frac{k^3}{(2\pi^2)} |z|^2 \right]^{1/2} = \left[ \frac{k^3}{(2\pi^2)} |y|^2 \right]^{1/2} = \frac{H}{2\pi} \quad (5.14)$$

is given exactly by the Hawking temperature [25], [34]. If inflation is not exactly DeSitter, this is approximately true in which case it is useful to interpret  $H \equiv H(k)$  as the Hubble parameter at the time that the comoving scale  $k^{-1}$  crossed the Hubble radius during inflation,  $k/(H\Omega) \sim 1$ , and

$$\mathcal{P}_\zeta(k) \sim \frac{9}{16\pi^2} H^4 / \left( \frac{\partial H}{\partial \phi} \right)^2. \quad (5.15)$$

For power-law inflation, we will compute this quantity exactly.

For the tensor modes, we define  $h_1(x)$  and  $h_2(x)$  as well as their corresponding power spectra, e.g.,

$$h_1(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} h_1(k), \quad (5.16)$$

with

$$\mathcal{P}_{h_1}(k) \equiv \frac{k^3}{2\pi^2} \int d^3x e^{-i\vec{k}\cdot\vec{x}} \langle h_1(x) h_1(0) \rangle = \frac{k^3}{2\pi^2} 2|y|^2. \quad (5.17)$$

Using the exact solution eq.(3.38) in the long-wavelength limit,  $u \rightarrow \infty$ , we find for power-law inflation that  $y$  and  $z$  are independent of time  $u$ :

$$|y| = |z| = \frac{2^{\nu-1}}{\sqrt{\pi}} \frac{\Gamma(\nu)}{\sqrt{p-1}} \left( \frac{k}{p-1} \right)^{\frac{-(3p-1)}{2(p-1)}}. \quad (u \rightarrow \infty) \quad (5.18)$$

The long-wavelength power spectra for power-law inflation are power-laws which are related to each other through the steepness parameter  $p$ , eq.(2.12a):

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\zeta(k_0) \left( \frac{k}{k_0} \right)^{n_s-1}, \quad (\text{long} - \text{wavelength}) \quad (5.19a)$$

$$\mathcal{P}_{h_1}(k) = \mathcal{P}_{h_2}(k) = \frac{4}{9p} \mathcal{P}_\zeta(k_0) \left( \frac{k}{k_0} \right)^{n_t-1}, \quad (\text{long} - \text{wavelength}) \quad (5.19b)$$

where, in the case of power-law inflation, the spectral indices,  $n_s, n_t$ , for scalar and tensor fluctuations actually coincide

$$n_s = n_t = 1 - \frac{2}{(p-1)}. \quad (5.20)$$

$\mathcal{P}_\zeta(k_0)$  is the value of the power spectrum in zeta at some fiducial wavenumber scale  $k_0$  which we will choose later. (It has been pointed out that inflation with a cosine potential can also yield a power-law fluctuation spectrum for scalar perturbations [51].)

## 2. Heating of the Universe

At the end of inflation, the scalar field typically rolls to the minimum of its potential where it oscillates and converts its energy into a thermal bath consisting of radiation and matter. Because the coupling of the scalar field to matter is not well understood, several possibilities can arise yielding either a low or high value of  $T_{max}$ , which we define to be the maximum temperature reached immediately after inflation. (Many models utilizing supersymmetry [52] give values,  $T_{max} \sim 10^8 \text{GeV}$ , whereas the Variable Planck Mass model [34]

is rather special in giving a very high result,  $T_{max} \sim 10^{15} \text{GeV}$ .) Fortunately, the amplitude of the metric at large wavelengths is indifferent to the uncertainty in  $T_{max}$ . (However, the present physical length of a comoving scale is indeed sensitive to the value of the maximum temperature). Numerical calculations for a simple phenomenological model describing coupling of a single scalar field to radiation and matter demonstrate that the amplitude of the metric fluctuation  $\zeta$  on a comoving slice remains constant [34]. Essentially a result of momentum conservation, the rest frame of the single scalar field (uniform  $\phi$ ) slice is coincident with the rest frame of radiation which is identical to the rest frame of the matter (uniform  $\chi$  slice) at wavelengths larger than the Hubble radius. Hence the fluctuations for structure formation arising from inflation are typically *adiabatic*. (If there are two scalar fields which are important during inflation, one may also produce isocurvature perturbations which we will not consider here; see, e.g., Sasaki and Yokoyama [53].) Moreover, the function  $y$  describing the tensor fluctuations in eqs.(3.30) and (4.12) is continuous during the heating process.

If we are only interested in large angle microwave background fluctuations produced in the cold-dark-matter model, we may thus equate the probability functionals on comoving time slices before and after heating of the Universe:

$$\mathcal{P}[\gamma_{ij}|u = u_{heat}] = \mathcal{P}[\gamma_{ij}|\chi = \chi_{heat} = 0]. \quad (5.21)$$

Hence for long-wavelength fields in the radiation and matter dominated eras, the power spectra for zeta as well as that for tensor perturbations are identical to those arising from inflation, eqs.(5.19a) and (5.19b). In other words, the scalar and tensor operators,  $\tilde{S}$  and  $\tilde{T}$  are continuous on a comoving time slice.

## B. Application of Sachs-Wolfe Formula

It remains useful to continue employing comoving slices of uniform  $\chi$  since the phase transition where radiation becomes uncoupled from matter occurs on a slice where the temperature is uniform,  $T = 4000K$ , which for adiabatic fluctuations, coincides with a uniform

$\chi$  slice at large wavelengths (see, e.g., ref. [37]). Hence by observing the phase transition, one has effectively *chosen* a very special time-hypersurface. The Sachs-Wolfe [48] formula yields the large angle temperature anisotropy from a line integral over the perturbation in the 3-metric  $h_{ij}$  computed in uniform  $\chi$  gauge (*comoving, synchronous gauge*):

$$\Delta T(x)/T = -\frac{1}{2}e^ke^l \int_{t_1}^{t_2} d\chi \frac{\partial h_{kl}}{\partial \chi}[\chi, x(\chi)] \quad (5.22)$$

The line integral traces the path  $x(\chi)$  of a photon path from the surface of last scattering to the present epoch;  $e^i$  is a unit vector giving the direction of the photon's velocity. For angles of interest to COBE ( $\alpha > 7^\circ$ ), we are concerned with those comoving scales that reenter the Hubble radius during the matter-dominated era.

### 1. Scalar Perturbations

In comoving synchronous gauge, the scalar part of the metric evolves according to

$$h_{ij}^{(s)}(\tau, x) = \frac{2}{3}\zeta(x)\delta_{ij} + \frac{1}{15}\tau^2 \zeta_{,ij}(x), \quad (\text{with } N = 1, N^i = 0). \quad (5.23)$$

(This may be derived most simply by using ref. [3], or by applying Sec.IV.) Here we have defined conformal time  $\tau \equiv \tau(\chi)$  to be

$$\tau = \int_0^\chi \frac{d\chi}{a(\chi)} = \frac{2}{H_0} \left( \frac{\chi}{\chi_0} \right)^{1/3}, \quad a(\chi) = \left( \frac{\chi}{\chi_0} \right)^{2/3}, \quad \chi_0 = \frac{2}{3H_0}, \quad (5.24)$$

where  $H_0$  is the present value of the Hubble parameter which we assume to be 50 km/s /Mpc which is consistent with measurements of the Sunyaev-Zel'dovich effect by Birkinshaw *et al* [54] (see also Lasenby [55]). In order to agree with our present units of measurement, we have normalized the scale factor  $a(\chi) \propto \Omega(\chi)$  to unity at the present epoch  $\chi_0$ . Integrating eq.(5.22) twice by parts and retaining only those boundary terms which are important for  $\alpha > 2^\circ$ , one finds that the large-angle temperature anisotropy,

$$\Delta T(x)/T = -\zeta(x)/15, \quad (5.25)$$

in a flat, matter-dominated Universe is proportional to the value of  $\zeta(x)$  on the surface of last scattering which is a sphere of radius  $R = |x| = 11,700\text{Mpc}$ . In this way, temperature fluctuations are a probe of scalar fluctuations from inflation.

Since we are concerned with temperature anisotropies measured on the celestial sphere, it is natural to employ a spherical harmonic expansion. A plane wave can be decomposed into orthogonal spherical harmonic functions,  $Y_{\ell m}$ , and spherical Bessel functions,  $j_\ell$ , through (see, e.g., ref. [56])

$$e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} i^\ell j_\ell(kr) Y_{\ell m}^*(\Omega_x) Y_{\ell m}(\Omega_k). \quad (5.26)$$

If  $\delta$  denotes the angle between  $\vec{k}$  and  $\vec{x}$ , then the addition theorem relates the spherical harmonics with the Legendre polynomial  $P_\ell(\cos\delta)$ :

$$P_\ell(\cos\delta) = \frac{4\pi}{(2\ell+1)} \sum_{m=-\ell}^{m=\ell} Y_{\ell m}^*(\Omega_x) Y_{\ell m}(\Omega_k). \quad (5.27)$$

Hence, the plane wave expansion

$$\zeta(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left( \frac{2\pi^2}{k^3} \mathcal{P}_\zeta(k) \right)^{1/2} \beta_3(k), \quad (5.28)$$

(which follows from the expression for the scalar power spectrum eq.(5.12) of a Gaussian random field) implies that the scalar contribution to the angular correlation function  $C_s(\alpha)$  can be expressed as a sum over the Legendre polynomials  $P_\ell(\cos\alpha)$  [57]:

$$C_s(\alpha) \equiv \langle \Delta T(x) \Delta T(x') \rangle_s = \sum_{\ell=0}^{\infty} \langle \Delta T_\ell^2 \rangle_s P_\ell(\cos\alpha), \quad (5.29)$$

$$\langle \Delta T_\ell^2 \rangle_s \equiv \left( \frac{T_\gamma}{15} \right)^2 (2\ell+1) \int_0^\infty \frac{dk}{k} \mathcal{P}_\zeta(k) j_\ell^2(kR) \quad (5.30a)$$

$$= A^2 \frac{(2\ell+1)}{5} \frac{\Gamma(\ell + (n_s - 1)/2)}{\Gamma(\ell + (5 - n_s)/2)} \frac{\Gamma((9 - n_s)/2)}{\Gamma((3 + n_s)/2)}, \quad (5.30b)$$

$$A^2 = T_\gamma^2 \mathcal{P}_\zeta(k_0) (k_0 R)^{1-n_s} \frac{\pi}{45} 2^{n_s-4} \frac{\Gamma(3 - n_s)}{\Gamma^2(2 - n_s/2)} \frac{\Gamma((n_s + 3)/2)}{\Gamma((9 - n_s)/2)} \quad (5.30c)$$

where  $\alpha$  is the angle between the two points,  $x$  and  $x'$ , on the surface of last scattering;  $T_\gamma = 2.736 \pm 0.017 K$  is the mean background temperature [58].  $\mathcal{P}_\zeta(k_0)$  was defined in eq.(5.19a), and we will assume that the fiducial wavenumber scale is  $k_0 = 10^{-4}\text{Mpc}$ . (Integrals of various combinations of Bessel functions may be found in Gradshteyn and Ryzhik [59]).

## 2. Tensor Perturbations

The derivation of microwave anisotropies from tensor perturbations is similar in principle to the scalar case. However it is technically more complicated because the angular correlation function obtains contributions from points within the surface of last scattering.

For primordial gravitational waves described by eq.(5.19b), the metric for the tensor modes evolves according to

$$h_{ij}^{(t)}(\chi, x) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left( \frac{2\pi^2}{k^3} \mathcal{P}_{h_1}(k) \right)^{1/2} \frac{3j_1(k\tau)}{(k\tau)} \sum_{a=1}^2 \beta_a(k) E_{ij}^{(a)}(k). \quad (5.31)$$

during the matter-dominated era. By applying a continuity argument after the exit of inflation to the radiation and matter dominated eras, we have determined the coefficients  $A$  and  $B = 0$  in eq.(4.14); we have also chosen to rewrite eq.(4.14) in terms of a spherical Bessel function  $j_1(k\tau)$  of order 1. Using the Sachs-Wolfe formula eq.(5.22), we find that the temperature anisotropy is

$$\Delta T/T = -\frac{1}{2} e^i e^j \int_{\tau_1}^{\tau_2} d\tau \int \frac{d^3k}{(2\pi)^3} \left( \frac{2\pi^2}{k^3} \mathcal{P}_{h_1}(k) \right)^{1/2} e^{i\vec{k}\cdot\vec{x}(\tau)} \frac{\partial}{\partial \tau} \left( \frac{3j_1(k\tau)}{(k\tau)} \right) \sum_{a=1}^2 \beta_a(k) E_{ij}^{(a)}(k), \quad (5.32)$$

and the resulting two-point correlation function is

$$\begin{aligned} \left\langle \frac{\Delta T(x)}{T} \frac{\Delta T(x')}{T} \right\rangle_t &= \frac{1}{4} e^i e^j e'^p e'^q \int_{\tau_1}^{\tau_2} d\tau \int_{\tau_1}^{\tau_2} d\tau' \int \frac{d^3k}{(2\pi)^3} \frac{2\pi^2}{k^3} \mathcal{P}_{h_1}(k) e^{i\vec{k}\cdot\vec{x}(\tau) - i\vec{k}\cdot\vec{x}'(\tau')} \\ &\quad \frac{\partial}{\partial \tau} \left( \frac{3j_1(k\tau)}{(k\tau)} \right) \frac{\partial}{\partial \tau'} \left( \frac{3j_1(k\tau')}{(k\tau')} \right) \sum_{a=1}^2 E_{ij}^{(a)}(k) E_{pq}^{(a)}(k), \end{aligned} \quad (5.33)$$

where

$$x^i = \left( \frac{2}{H_0} - \tau \right) e^i, \quad x'^i = \left( \frac{2}{H_0} - \tau' \right) e'^i, \quad (5.34)$$

describes the paths of the photons, with  $e^i = x^i/|x|$ ,  $e'^i = x'^i/|x'|$ . The values of conformal time at the present epoch and at the time of decoupling,  $(1+a)^{-1} \sim 1300$ , are, respectively,  $\tau_2 = 2/H_0$  and  $\tau_1 = 2/(H_0\sqrt{1300})$ . This expression may be simplified by noting that

$$\sum_{a=1}^2 E_{ij}^{(a)}(k) E_{pq}^{(a)}(k) = \left[ \tilde{\delta}_{ip} \tilde{\delta}_{jq} + \tilde{\delta}_{iq} \tilde{\delta}_{jp} - \tilde{\delta}_{ij} \tilde{\delta}_{pq} \right], \quad \text{where} \quad \tilde{\delta}_{ij} = \delta_{ij} - k_i k_j / k^2, \quad (5.35)$$

so that

$$e^i e^j e'^p e'^q \sum_{a=1}^2 E_{ij}^{(a)}(k) E_{pq}^{(a)}(k) = 2 (\cos\alpha - \cos\theta \cos\theta')^2 - (1 - \cos^2\theta) (1 - \cos^2\theta'). \quad (5.36)$$

Here  $\theta$  denotes the angle between  $\vec{k}$  and  $\vec{x}$ , and analogously for  $\theta'$ , and once again,  $\alpha$  is the angle between  $x$  and  $x'$ . The various factors of  $\cos\theta$  (and  $\cos\theta'$ ) may be removed by several applications of the identity,

$$\cos\theta P_\ell(\cos\theta) = \frac{(\ell+1)}{(2\ell+1)} P_{\ell+1}(\cos\theta) + \frac{\ell}{(2\ell+1)} P_{\ell-1}(\cos\theta), \quad (5.37)$$

in conjunction with the plane wave decomposition into Legendre polynomials, eqs.(5.26) and (5.27). The angular integrations in eq.(5.33) may be performed, and once again the angular correlation function can be written in a series of Legendre polynomials,

$$C_t(\alpha) = \sum_{\ell=0}^{\infty} \langle \Delta T_\ell^2 \rangle_t P_\ell(\cos\alpha), \quad (5.38)$$

$$\langle \Delta T_\ell^2 \rangle_t = \frac{9}{4} T_\gamma^2 (\ell-1)\ell(\ell+1)(\ell+2)(2\ell+1) \quad (5.39)$$

$$\int_0^{2k_{max}/H_0} dw w \mathcal{P}_{h_1} \left( \frac{H_0 w}{2} \right) \left\{ \int_0^{1-(H_0 \tau_1)/2} dv \quad (5.40)$$

$$\left[ 3 \frac{\cos(w(1-v))}{w^3(1-v)^3} + \frac{\sin(w(1-v))}{w^2(1-v)^2} - 3 \frac{\sin(w(1-v))}{w^4(1-v)^4} \right] \quad (5.41)$$

$$\left[ \frac{j_{\ell+2}(wv)}{(2\ell+1)(2\ell+3)} + 2 \frac{j_\ell(wv)}{(2\ell-1)(2\ell+3)} + \frac{j_{\ell-2}(wv)}{(2\ell+1)(2\ell-1)} \right] \Bigg\}^2 \quad (5.42)$$

which was derived by Abbott and Wise [45] (see also Starobinsky [60]).  $k_{max} = 10^{-2} \text{Mpc}^{-1}$  corresponds to the Hubble radius at matter-radiation equality. For angular scales that are measured by COBE ( $\alpha > 7^\circ$ ), the precise value of  $k_{max}$  is not that important, since the spherical Bessel functions  $j_\ell(\omega v)$  provide a cutoff.

Since the scalar and tensor contributions are independent Gaussian random fields, they add in quadrature

$$\langle \Delta T_\ell^2 \rangle = \langle \Delta T_\ell^2 \rangle_s + \langle \Delta T_\ell^2 \rangle_t. \quad (5.43)$$

In Fig.(3), we have computed the relative contribution of the tensor component  $\langle \Delta T_\ell^2 \rangle_t / (\langle \Delta T_\ell^2 \rangle_s + \langle \Delta T_\ell^2 \rangle_t)$  for various values of the spectral index  $n \equiv n_s = n_t = 1 - 2/(p-1)$ . For smaller values of  $n_s$ , it increases quite dramatically.

### C. Recent Observations

The COBE DMR team has recently analyzed their 2-year data set [28]. At the 68% confidence level, they find that the spectral index [29] for scalar perturbations is  $n_s = 1.10 \pm 0.32$ , which is consistent with the simplest models of inflation models which yield  $n_s < 1$ . They also determine that the root-mean-square temperature anisotropy with dipole removed is  $\sigma_{sky}(10^0) = 30.5 \pm 2.7 \mu K$  at the same level of confidence. The latter quantity,

$$\sigma_{sky}^2(10^0) = \sum_{l=2}^{\infty} \langle \Delta T_l^2 \rangle \exp \left[ -l(l+1)/13.5^2 \right], \quad \sigma_{sky}(10^0) = 30.5 \pm 2.7 \mu K,$$

is computed using the sum of the scalar and tensor fluctuations eq.(5.43). It determines the arbitrary normalization factor appearing in long-wavelength power spectra, eqs.(5.19a,b). The exponential factor corresponds to a Gaussian window function with full width at half maximum of  $10^0$ .

However, COBE by itself cannot discriminate between tensor and scalar fluctuations. One needs an additional experiment to measure the scalar perturbations. Two proposals have been suggested using either: (1) galaxy clustering data [26] or (2) intermediate microwave background experiments  $1^0 < \alpha < 2^0$  [61]. We shall discuss only the first proposal here since there are large variations in the intermediate angle observations [62] – [65].

In Fig.(4), we have computed the power-spectra for zeta that arises from the various power-law inflation models, assuming that they account for COBE's measurement of  $\sigma_{sky}(10^0)$ . In the limit that  $n_s \rightarrow 1$  ( $p \rightarrow \infty$ ) gravitational waves do not contribute to COBE's signal, and the power spectrum for zeta is the flat, Zel'dovich spectrum. As  $n_s$  decreases, gravitational waves are significant, leaving a smaller contribution for the scalar perturbation to  $\sigma_{sky}(10^0)$ . Hence at the fiducial wavenumber  $k_0 = 10^{-4} \text{Mpc}^{-1}$  (scales probed by COBE),  $\mathcal{P}_\zeta(k_0)$  decreases as  $n_s$  decreases. Moreover, the slope of the power spectrum becomes more negative as  $n_s$  decreases.

In comoving synchronous gauge (uniform  $\chi$ ),  $\rho\gamma^{1/2}$  is independent of time and the linear density perturbation at early times is

$$\delta(\tau, x) = (\rho - \bar{\rho}) / \bar{\rho} = -\frac{\tau^2}{30} \zeta_{,l}^l(x), \quad (\text{early times}) \quad (5.44)$$

which may be derived from the expression for the metric eq.(5.23). In Fourier space this yields,

$$\delta(\tau, k) = \frac{k^2 \tau^2}{30} \zeta(k). \quad (5.45)$$

During the radiation-dominated era, density perturbations oscillate and they damp because of the Hubble expansion. This effect is described by the transfer function  $T(k)$ , so that in the matter-dominated era the linear density perturbation is given by

$$\delta(\tau, k) = \frac{k^2 \tau^2}{30} T(k) \zeta(k). \quad (5.46)$$

This requires an assumption for the dark matter, and we have adopted the cold-dark-matter transfer function [57].

In Fig.(5), we show power-spectra for the density perturbation,

$$\mathcal{P}_\delta(\tau, k) \equiv \frac{k^3}{2\pi^2} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \langle \delta(\tau, x) \delta(\tau, 0) \rangle, \quad (5.47)$$

arising from power-law inflation. The bold line depicts the power-spectrum

$$\mathcal{P}_\delta^{(obs)}(k) = \frac{2}{\pi} \sin\left(\frac{\pi\gamma}{2}\right) \Gamma(2-\gamma) (kr_0)^\gamma \quad (5.48)$$

for the observed galaxy-galaxy correlation function  $\xi_{gg}(r) = (r/r_0)^{-\gamma}$  where  $r_0 = 10$  Mpc,  $\gamma = 1.8$ , and  $\Gamma$  is the gamma function. The corresponding biasing parameter  $b_\rho$  is found by computing the mass fluctuation on a scale of 16 Mpc:

$$\left\langle \left( \frac{\Delta M}{M}(r = 16 Mpc) \right)^2 \right\rangle = 1/b_\rho^2. \quad (5.49)$$

For  $n_s = 1, 0.95, 0.9, 0.85, 0.8, 0.7, 0.5$ , we compute the biasing parameter to be  $b_\rho = 0.82, 1.06, 1.34, 1.65, 2.0, 2.88, 5.6$ . In order to be consistent with the biased galaxy formation, we insist that  $b_\rho < 2$  and  $0.8 < n_s < 1$  otherwise there are not enough fluctuations to seed galaxies. As a result, for power-law inflation no more than 50% of COBE's signal can arise from gravitational waves.

Previously, it had been suggested that power-law inflation with  $n_s = 0.5$  [66] could account for the APM (Automatic Plate machine) survey [67] which demonstrated more power than predicted by the standard cold-dark-matter model. However, Salopek [26] pointed out that the production of gravitational radiation, which was neglected in the previous calculation, could be quite significant for power-law inflation. He was able to rule out this promising model for large scale power since it is essential that  $n_s > 0.8$ . (For a careful discussion of statistical limits on the spectral index  $n_s$  using only COBE data, consult Kurki-Suonio and Mathews [68]).

## VI. CONCLUSIONS

Hamilton-Jacobi methods are a cornerstone of modern theoretical physics, and they may be profitably applied to numerous problems in cosmology. For example, they have been successfully employed in deriving the Zel'dovich approximation and its higher generalizations from general relativity [7] – [9]. Various researchers have employed HJ methods in an attempt to recover the inflaton potential from cosmological observations [41]. Moreover, they can be used to construct inflationary models that yield non-Gaussian primordial fluctuations [4]; such models could possibly resolve the problem of large scale structure [69]. Here we have given a careful and detailed computation of the galaxy-galaxy correlation function and large-angle microwave background fluctuations arising from power-law inflation, which is the most interesting model involving gravitational radiation. We find that the resulting spectral index for scalar perturbations must satisfy  $n_s > 0.8$ , otherwise the production of gravitational radiation is excessive, and there are not enough fluctuations to seed galaxies.

Our analysis is greatly facilitated by the fact that a choice of the time-hypersurface is not required in the computation of the probability functional during the inflationary epoch. Field-space diagrams are useful in visualizing a hypersurface transformation. However in the end when one compares with observations one typically assumes a particular choice of gauge. For large-angle microwave background fluctuations which are computed using the

Sachs-Wolfe formula, comoving synchronous gauge is preferred.

Our line integral formulation of the HJ equation (3.15) goes a long way in illuminating the role of time in semi-classical general relativity. Different time-hypersurface choices correspond to different choices of contours in superspace. Provided spatial gauge invariance is maintained, they all yield the same result for the generating functional.

A complete quantum formulation of the gravitational field is still lacking. String theory is a possible candidate, and its applications to cosmology are currently being investigated [70], [71]. Our aim is more modest in that we have restricted ourselves to the semi-classical theory of Einstein gravity which is nonetheless adequate in describing various quantum gravitational phenomena including graviton fluctuations beginning initially in the ground state [26]. We hence follow in spirit the historical development of the theory of atomic spectra. Before the development of the quantum theory in 1926, the semi-classical theory of Bohr and Sommerfeld provided a useful although imperfect description of various atoms.

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## REFERENCES

- [1] C. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (New York: Freeman, 1973).
- [2] S. Tomonaga, in *Quantum Electrodynamics*, ed. J. Schwinger (Dover, New York, 1958); J. Schwinger, *ibid.*.
- [3] D.S. Salopek and J.M. Stewart, *Class. Quantum Grav.* **9**, 1943 (1992).
- [4] D.S. Salopek, **43**, 3214 (1991); *ibid* **45**, 1139 (1992); D.S. Salopek and J.R. Bond, *Phys. Rev. D* **42**, 3936 (1990).
- [5] J. Parry, D.S. Salopek and J.M. Stewart, *Phys. Rev. D* **49**, 2872 (1994).
- [6] Y.B. Zel'dovich, A & A, **5**, 84 (1970).
- [7] K.M. Croudace, J. Parry, D.S. Salopek and J.M. Stewart, *Ap. J.* **423**, 22 (1994).
- [8] D.S. Salopek, J.M. Stewart, K.M. Croudace and J. Parry, in Proc. 37th Yamada Conference, June 8-12, 1993, Tokyo, Japan, ed. K. Sato (Universal Academic Press, Inc. 1994).
- [9] D.S. Salopek, J.M. Stewart and K.M. Croudace, *Mon. Not. Roy. Astr. Soc.* (in press, 1994).
- [10] F. Moutarde, J.-M. Alimi, F.R. Bouchet, R. Pellat and A. Ramani, *Astrophys. J.* **382**, 377 (1991).
- [11] T. Buchert and J. Ehlers, *Mon. Not. Roy. Astr. Soc.* **264**, 375 (1993).
- [12] A.O. Barvinsky and G.A. Vilkovisky, *Nucl. Phys. B* **333**, 471 (1990).
- [13] Nuffield Workshop, *The Very Early Universe*, Cambridge, June 21- July 9, 1982, ed. G.W. Gibbons, S.W. Hawking and S.T.C. Siklos (Cambridge University Press, 1983).
- [14] J.M. Bardeen, P.J. Steinhardt and M.S. Turner, *Phys. Rev. D* **28**, 679 (1983).
- [15] A. Guth and S.Y. Pi, *Phys. Rev. Lett.* **49**, 1110 (1982).

- [16] S.W. Hawking, *Phys. Lett.* **115B**, 295 (1982).
- [17] A.A. Starobinsky, *Phys. Lett.* **117B**, 175 (1982).
- [18] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, *Phys. Rep.* **215**, No.5 & 6, 203 (1992); V.F. Mukhanov, *Zh. Eksp. Tor. Fiz.* **94**, 1 (1988).
- [19] A.A. Starobinsky, in *Current Topics in Field Theory, Quantum Gravity, and Strings*, ed. H.T. de Vega and N. Sanchez (Springer: New York, 1986), p.107.
- [20] A.D. Linde, *Phys. Lett. B* **175**, 395 (1986).
- [21] J.M. Stewart, *Class. Quantum Grav.* **8**, 909 (1991).
- [22] D.S. Salopek and J.R. Bond, *Phys. Rev. D* **43**, 1005 (1991).
- [23] D. Boyanovsky, H.J. DeVega and R. Holman, *Phys. Rev. D* **49**, 2769 (1994).
- [24] A. Linde, D. Linde and A. Mezhlumian, *Phys. Rev. D* **49**, 1783 (1994).
- [25] T.S. Bunch, and P.C.W. Davies, *Proc. R. Soc. Lond. A.* **360**, 117 (1978).
- [26] D.S. Salopek, *Phys. Rev. Lett.*, **69**, 3602 (1992); *Inter. J. Mod. Physics D* **3**, 257 (1994); *Proc. of the International School of Astrophysics, "D. Chalonge", Second Course: Current Topics in Astrofundamental Physics, Erice, Italy, September 6-13, 1992*, p. 23, ed. N. Sanchez and A. Zichichi (World Scientific Publishers, 1993).
- [27] E.L. Wright *et al*, *Astrophys. J. Lett.* **396**, L13 (1992); G.F. Smoot *et al*, *Astrophys. J. Lett.* **396**, L1 (1992).
- [28] C.L. Bennett *et al*, COBE preprint (1994).
- [29] K.M. Gorski *et al*, COBE preprint (1994).
- [30] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Academic, New York, 1964).
- [31] A. Peres, *Nuovo Cim.* **26**, 53, (1962).

- [32] K. Kuchař, *J. Math. Phys.* **11**, 3322 (1970).
- [33] J.J. Halliwell and S.W. Hawking, *Phys. Rev. D* **31**, 1777 (1985).
- [34] D.S. Salopek, J.R. Bond and J.M. Bardeen, *Phys. Rev. D* **40**, 1753 (1989).
- [35] A.O. Barvinsky, *Physics Reports* **230**, 5 & 6, 237 (1993); *Phys. Lett.* **B241**, 201 (1990);  
A.O. Barvinsky and A.Y. Kamenshchik, *Phys. Lett. B* (in press, 1994).
- [36] D.S. Salopek, *Phys. Rev. D* **46**, 4373 (1992).
- [37] D.S. Salopek and J.M. Stewart, *Phys. Rev. D* **47**, 3235 (1993).
- [38] B.F. Schutz, *Phys. Rev. D* **4**, 3559 (1971).
- [39] F. Lucchin and S. Matarrese, *Phys. Rev. D* **32**, 1316 (1985).
- [40] D. La and P.J. Steinhardt, *Phys. Rev. Lett.* **62**, 376 (1989); E.W. Kolb, D.S. Salopek and  
M.S. Turner, *Phys. Rev. D* **42**, 3925 (1990); N. Deruelle, C. Gundlach and D. Langlois,  
*Phys. Rev. D* **45**, 3301 (1992).
- [41] E.J. Copeland, E.W. Kolb, A.R. Liddle and J.E. Lidsey, *Phys. Rev. D* **48**, 2529 (1993);  
E.W. Kolb and S.L. Vadas, *Phys. Rev. D* (in press, 1994).
- [42] G.L. Comer, N. Deruelle, D. Langlois and J. Parry, *Phys. Rev. D* **49**, 2759 (1994).
- [43] V. Moncrief and C. Teitelboim, *Phys. Rev. D* **6**, 966 (1972).
- [44] D.S. Salopek, J.M. Stewart and J. Parry, *Phys. Rev. D* **48**, 719 (1993).
- [45] L.F. Abbott and M.B. Wise, *Nucl. Phys. B* **244**, 541 (1984).
- [46] D. Lyth and E. Stewart, *Phys. Lett.* **B274**, 168 (1992).
- [47] N.D. Birrell and P.C.W. Davies, *Quantum fields in Curved Space*, (Cambridge University  
Press, 1989).
- [48] R.K. Sachs and A.M. Wolfe, *Astrophys. J.* **147**, 73 (1967).

- [49] D. Langlois, Observatoire de Paris-Meudon preprint (1993).
- [50] J. Hwang, *Phys. Rev. D* **48**, 3544 (1993).
- [51] F.C. Adams, J.R. Bond, K. Freese, J.A. Frieman and A.V. Olinto, *Phys. Rev. D* **47**, 426 (1993).
- [52] B.A. Campbell, S. Davidson and K.A. Olive, *Nucl. Phys. B* **399**, 111 (1993).
- [53] M. Sasaki and J. Yokoyama, *Phys. Rev. D* **44**, 970 (1991).
- [54] M. Birkinshaw, J.P. Hughes and K.A. Arnaud, *Ap. J.* **379**, 466 (1991).
- [55] A. N. Lasenby, Proc. of the International School of Astrophysics, “D. Chalonge”, Second Course: Current Topics in Astrofundamental Physics, Erice, Italy, September 6-13, 1992, p. 209, ed. N. Sanchez and A. Zichichi (World Scientific Publishers, 1993).
- [56] J.D. Jackson, *Classical Electrodynamics*, (2nd Edition, John Wiley & Sons, 1975).
- [57] P.J.E. Peebles, *Astrophys. J.* **263**, L1 (1982); J.R. Bond and G. Efstathiou, *Astrophys. J. Lett.* **285**, L45 (1984); *Mon. Not. R. Astr. Soc.* **226**, 655 (1987).
- [58] H.P. Gush, M. Halpern and E.H. Wishnow, *Phys. Rev. Lett.* **65**, 537 (1990); J.C. Mather *et al*, *Ap.J.* 354, 137 (1990).
- [59] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integral, Series and Products*, (Academic Press, 1980)
- [60] A. A. Starobinsky, *Sov. Astron. Lett.* **11**, 133 (1985).
- [61] R.L. Davis, H.M. Hodges, G.F. Smoot, P.J. Steinhardt and M.S. Turner, *Phys. Rev. Lett.* **69**, 1856 (1992); R. Crittenden, J.R. Bond, R.L. Davis, G. Efstathiou and P.J. Steinhardt, *Phys. Rev. Lett.* **71**, 324 (1993).
- [62] J. Schuster, T. Gaier, J. Gundersen, P. Meinhold, T. Koch, M. Seiffert, C.A. Wuensche and P. Lubin, *Astrophys. J. Lett.* **412**, L47 (1993).

- [63] J.O. Gundersen, A.C. Clapp, M. Devlin, W. Holmes, M.L. Fischer, P.R. Meinhold, A.E. Lange, P.M. Lubin, P.L. Richards and G.F. Smoot, *Astrophys. J. Lett.* **413**, L1 (1993).
- [64] E.J. Wollack, N.C. Jarosik, C.B. Netterfield, L.A. Page and D. Wilkinson, *Astrophys. J. Lett.* **419**, L49 (1993).
- [65] E.L. Wright, G.F. Smoot, C.L. Bennett and P.M. Lubin, *Astrophys. J.* (in press, 1994).
- [66] A. Liddle, D. Lyth and W. Sutherland, *Phys. Lett. B* **279**, 244 (1992).
- [67] S.J. Maddox, G. Efstathiou, W.J. Sutherland and J. Loveday, *Mon. Not. Roy. Astr. Soc.* **242**, 43P (1990).
- [68] H. Kurki-Suonio and G.J. Mathews, Lawrence Livermore preprint (1994).
- [69] L. Moscardini, S. Borgani, P. Coles, F. Lucchin, S. Matarrese, A. Messina, M. Plionis, *Astrophys. J.* **413**, 55 (1993).
- [70] M. Gasperini, N. Sanchez and G. Veneziano, *Nucl. Phys. B* **364**, 365 (1991); *Int. J. Mod. Phys. A*, Vol.6, No.21, 3853 (1991).
- [71] H.J. DeVega and N. Sanchez, *Phys. Rev. D* **47**, 3394 (1993); B.A. Campbell, N. Kaloper and K.A. Olive, *Phys. Lett.* **B277**, 265 (1992).

## VII. FIGURE CAPTIONS

**Fig.(1):** In the long-wavelength limit, the evolution of the scalar field and the metric are shown for inflation with an exponential potential. For each spatial point corresponds a trajectory (which is a straight line). Slicing this diagram in a particular direction would represent making a time-hypersurface choice.  $\Delta\phi(\alpha)$  denotes the scalar field fluctuation on a time-hypersurface of uniform  $\alpha = (\ln \gamma)/6$ .  $\Delta\alpha(\phi)$  refers to the metric fluctuation on a time-hypersurface of uniform  $\phi$ . These perturbations are related to each other through the slope of the  $\alpha - \phi$  trajectories.

**Fig.(2):** In the long-wavelength limit, the evolution of the dust field and the metric are shown when a cosmological constant is present.  $\Delta\alpha(\chi)$  refers to the metric fluctuation on a time-hypersurface of uniform  $\chi$  (comoving, synchronous gauge).  $\Delta\chi(\alpha)$  denotes fluctuation of the dust field on a time-hypersurface of uniform  $\alpha$ . The hypersurface transformation relating the two is more complicated than in Fig.(1) because here the trajectories are curved.

**Fig.(3):** The gravity wave contribution to large angle  $\Delta T/T$  can be dominant for power-law inflation which employs an exponential potential,  $V(\phi) = V_0 \exp[-\sqrt{\frac{2}{p}}\phi]$ . As a function of the spherical harmonic  $\ell$ , the relative contribution of the gravity waves  $\langle \Delta T_\ell^2 \rangle_t$  to the total contribution  $\langle \Delta T_\ell^2 \rangle_s + \langle \Delta T_\ell^2 \rangle_t$  is plotted for various potential parameters  $n_s = 1 - 2/(p-1)$ . If one normalizes to COBE, then  $n_s > 0.8$  is required to give fluctuations large enough to produce galaxies.

**Fig.(4):** Primordial scalar perturbations of the metric are described by the function  $\zeta$ . The fluctuation spectra for zeta are shown for various choices of the spectral index  $n_s$  arising from power-law inflation. They have been normalized using COBE's 2-yr data set.

**Fig.(5):** For the present epoch, the power spectra for the linear density perturbation  $\delta$  in comoving synchronous gauge are shown. The dark line depicts the observed two-point correlation function describing galaxy clustering. If there is no biasing,  $n_s = 0.9$  gives a good fit to the observed data near  $k = 10^{-1} Mpc^{-1}$ . In order that there be enough fluctuations to seed galaxies, one requires that the biasing parameter  $b_\rho$  be less than 2 which implies that

$0.8 < n_s < 1$ . As a result, for power-law inflation, at most 50% of COBE's signal may arise from gravitational waves.

FIGURES

FIELD-SPACE DIAGRAM: SCALAR FIELD + GRAVITY

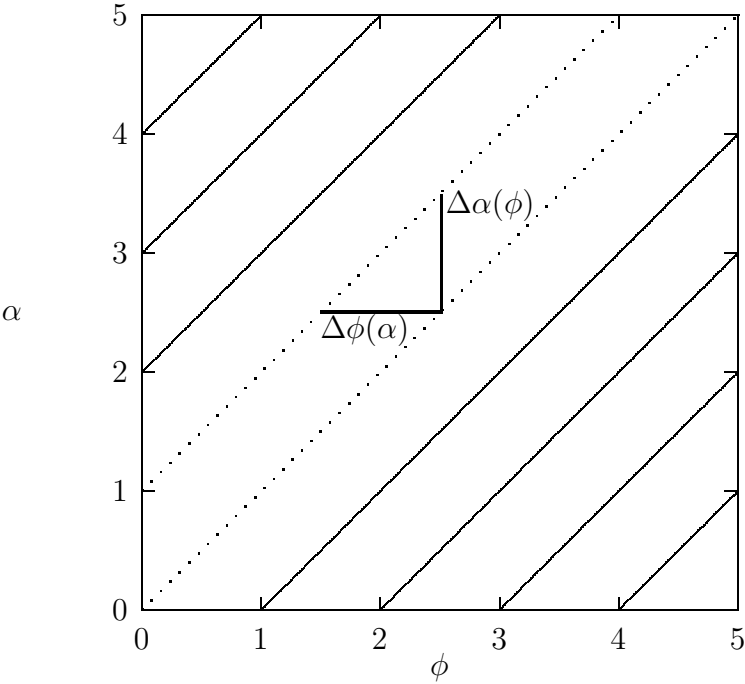


Fig.(1)

FIELD-SPACE DIAGRAM: DUST FIELD + GRAVITY

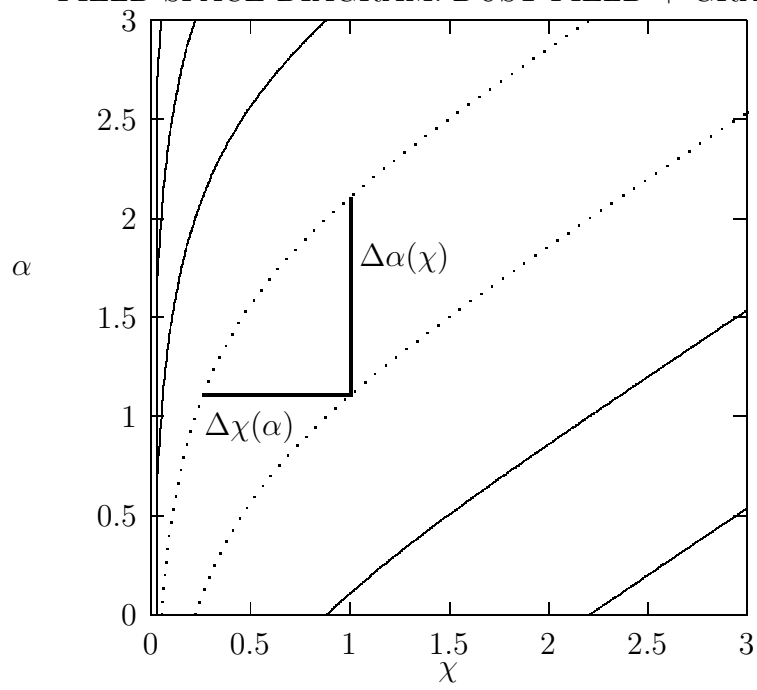


Fig.(2)

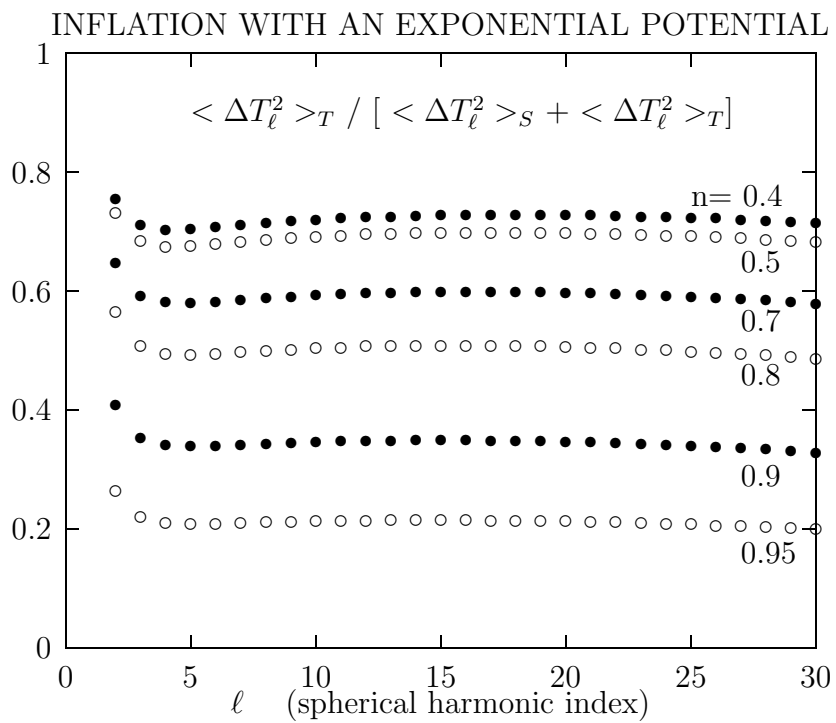


Fig.(3)

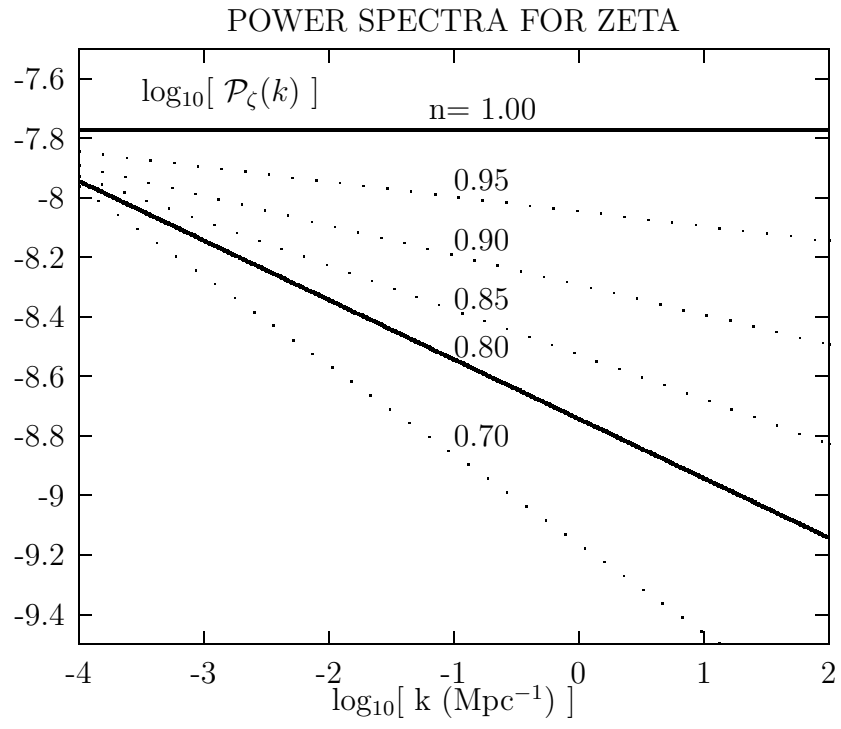


Fig.(4)

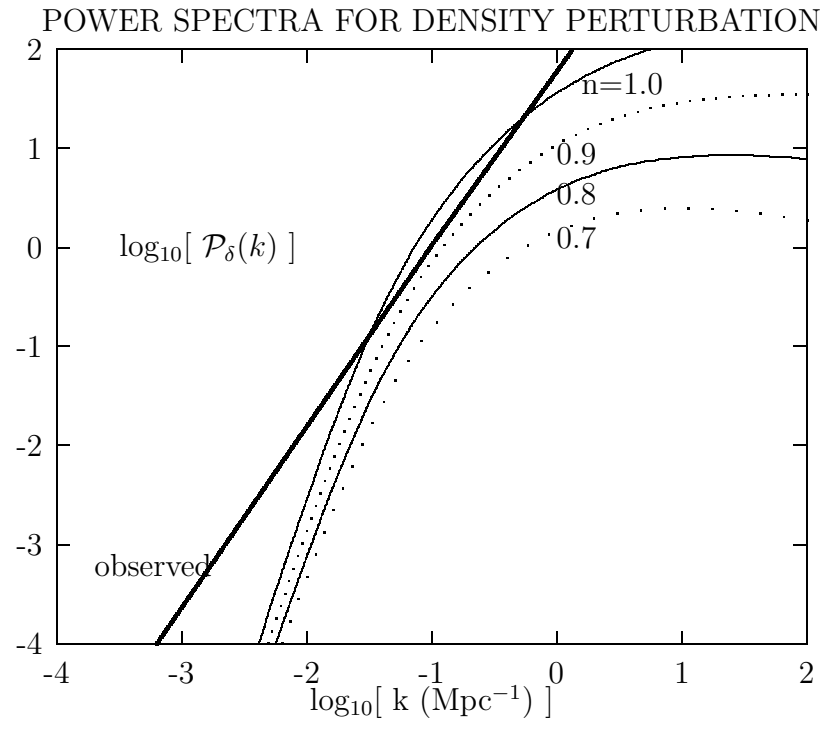


Fig.(5)